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Simple Cuntz–Pimsner rings [☆]

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Necessary and sufficient conditions for when every non-zero ideal in a relative Cuntz–Pimsner ring contains a non-zero graded ideal, when a relative Cuntz–Pimsner ring is simple, and when every ideal in a relative Cuntz–Pimsner ring is graded, are given. A “Cuntz–Krieger uniqueness theorem” for relative Cuntz–Pimsner rings is also given and condition (L) and condition (K) for relative Cuntz–Pimsner rings are introduced.
As applications of these results, a uniqueness result for the Toeplitz algebra of a directed graph and characterizations of when crossed products of a ring by a single automorphism and fractional skew monoid rings of a single corner isomorphism are simple, are obtained.

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1. Introduction

In [5], the two first named authors introduced the notion of a relative Cuntz–Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ as an algebraic analogue of (relative) Cuntz–Pimsner C^*-algebras (see for example [12, 15,7,9]). The class of relative Cuntz–Pimsner rings unifies and generalizes many interesting classes of (associative, but not necessarily unital) rings, for instance Leavitt path algebras (see for example [1,2,19]), crossed products of a ring by a single automorphism (also called a skew group ring, see for example [11] and [14]) and fractional skew monoid rings of a single corner isomorphism (see [3]).

Each relative Cuntz–Pimsner ring comes with a $\mathbb{Z}$-grading. In [5], a complete description of all the graded ideals (i.e. the ideals that are compatible with the above mentioned $\mathbb{Z}$-grading) of an arbitrary relative Cuntz–Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ is given. The purpose of this paper is to study the non-graded ideals of such a relative Cuntz–Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$. Although we do not reach a complete description of all (graded or non-graded) ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$, we do find necessary and sufficient conditions for when every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal (Theorem 4.2), when every ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ is graded (Theorem 6.2), and when $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple (Theorem 7.3). We also give a "Cuntz–Krieger uniqueness theorem" for $\mathcal{O}_{(P,Q,\psi)}(J)$ (Theorem 5.2) and introduce condition (L) (Definition 4.1) and condition (K) (Definition 6.1) for relative Cuntz–Pimsner rings. These results and definitions are generalizations of similar results and definitions about Leavitt path algebras given in [19], and analogues of similar results and definitions given in the C^*-algebraic setting for graph C^*-algebras (see for example [16]), ultragraph C^*-algebras (see [18]), topological graph C^*-algebras (see [8]), and (relative) Cuntz–Krieger algebras of finitely aligned higher rank graphs (see for example [17]).

It is worth pointing out that in the C^*-algebraic setting, analogues of Theorems 4.2, 5.2, 6.2 and 7.3 do not exist in the generality used in this paper. It does not seem unreasonable to believe that it should be possible to obtain such analogues, but a different approach than the one used in this paper seems to be needed because the proof of the pivotal Proposition 3.8 does not work in the C^*-algebraic setting.

A motivation for developing the results presented in this paper were to generalize the result obtained in [19] from the setting of Leavitt path algebras to the setting of relative Cuntz–Pimsner rings. We will throughout the paper illustrate our results by applying them to Leavitt path algebras and thereby recover many of the results of [19]. We will also introduce the Toeplitz algebra of a directed graph as an extension of the corresponding Leavitt path algebra and present a uniqueness theorem for this algebra. This uniqueness theorem is the algebraic analogue of the uniqueness theorem for the Toeplitz C^*-algebra of a directed graph given in [6, Theorem 4.1].

We will finish the paper by applying the theory developed in the paper to crossed products of a ring by a single automorphism and fractional skew monoid rings of a single corner isomorphism, and thereby obtain characterizations of when these algebras are simple.

1.1. Contents

In Section 2 we recall the definition of relative Cuntz–Pimsner rings and some further definitions and results from [5] and establish notation which will be used throughout this paper. In Section 2, we also recall how Leavitt path algebras (see for example [1,2,19]) are examples of relative Cuntz–Pimsner rings. We will throughout the paper return to this example to illustrate our results, and thereby recover some of the results of [19]. Section 3 contains some preliminary results and the pivotal Proposition 3.8 on which the rest of the paper depends on. In Section 4, condition (L) is introduced (Definition 4.1), and sufficient and necessary conditions for when every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal are given (Theorem 4.2). Section 5 contains the Cuntz–Krieger uniqueness theorem (Theorem 5.2). In Section 6, condition (K) is introduced (Definition 6.1), and sufficient and necessary conditions for when every ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ is graded are given (Theorem 6.2), and in Section 7 sufficient and necessary conditions for when $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple are given (Theorem 7.3). In Section 8, the case when $J=0$ and $\mathcal{O}_{(P,Q,\psi)}(J)$ is the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)}$ of (P,Q,ψ) is considered. In addition, the Toeplitz algebra of a directed is introduced and an algebraic analogue of the uniqueness theorem [6, Theorem 4.1] for Toeplitz C^*-algebra of a directed graph is given. Finally, in Section 9 we illustrate the results obtained in the paper by applying them to crossed products of a ring by a single automorphism and fractional skew monoid rings of a single corner isomorphism, and thereby obtain characterizations of when these algebras are simple.

2. Relative Cuntz–Pimsner rings

We begin by recalling some definitions and results from [5] and establish notation which will be used throughout this paper.

In this paper every ideal will be a two-sided ideal. The set of integers will be denoted by $\mathbb{Z}$, the set of positive integers will be denoted by $\mathbb{N}$ and the set of non-negative integers will be denoted by $\mathbb{N}_0$.

Throughout this paper R will denote a fixed (associative, but not necessarily unital) ring, and (P, Q, ψ) will denote a fixed R-*system*, i.e., P and Q are R-bimodules, and ψ is an R-bimodule homomorphism from $P \otimes Q$ (the R-balanced tensor product of P and Q) to R, cf. [5, Definition 1.1]. Recall from [5, Definition 7.1] that an ideal I in R is ψ-*invariant* if $\psi(px \otimes q) \in I$ for $p \in P$, $x \in I$ and $q \in Q$.

A *covariant representation* of the R-system (P, Q, ψ) is a quadruple (S', T', σ', B) satisfying:

(i) B is a ring,
(ii) $S' : P \to B$ and $T' : Q \to B$ are additive maps,
(iii) $\sigma' : R \to B$ is a ring homomorphism,
(iv) $S'(pr) = S'(p)\sigma'(r)$, $S'(rp) = \sigma'(r)S'(p)$, $T'(qr) = T'(q)\sigma'(r)$ and $T'(rq) = \sigma'(r)T'(q)$ for $p \in P$, $q \in Q$ and $r \in R$,
(v) $\sigma'(\psi(p \otimes q)) = S'(p)T'(q)$ for $p \in P$ and $q \in Q$,

cf. [5, Definition 1.2].

For $p \in P$ and $q \in Q$ let $\theta_{q,p}$ denote the right R-module homomorphism from Q to Q given by $\theta_{q,p}(x) = q\psi(p \otimes x)$ for $x \in Q$, and let $\theta_{p,q}$ denote the left R-module homomorphism from P to P given by $\theta_{p,q}(y) = \psi(y \otimes q)p$ for $y \in P$. As in [5, Definition 2.2], we let $\mathcal{F}_P(Q)$ be the set of right R-module homomorphism from Q to Q which can be written as a finite R-linear combination of elements from $\{\theta_{q,p} \mid q \in Q,\ p \in P\}$, and we let $\mathcal{F}_Q(P)$ be the set of left R-module homomorphism from P to P which can be written as a finite R-linear combination of elements from $\{\theta_{p,q} \mid q \in Q,\ p \in P\}$. We will throughout this paper assume that our R-system (P, Q, ψ) satisfies *condition* (*FS*), i.e., we assume that there for any finite set $\{q_1, \dots, q_n\} \subseteq Q$ and any finite set $\{p_1, \dots, p_m\} \subseteq P$ exist $\Theta \in \mathcal{F}_P(Q)$ and $\Psi \in \mathcal{F}_Q(P)$ such that $\Theta(q_i) = q_i$ and $\Psi(p_j) = p_j$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$, cf. [5, Definition 3.4]. It follows from [5, Proposition 3.11] that if (S', T', σ', B) is a covariant representation of the R-system (P, Q, ψ), then there is a unique ring homomorphism $\pi_{T',S'} : \mathcal{F}_P(Q) \to B$ satisfying $\pi_{T',S'}(\theta_{p,q}) = S'(p)T'(q)$ for all $p \in P$ and $q \in Q$.

As in [5, Definition 3.10], we let Δ denote the ring homomorphism from R to the ring of right R-module homomorphisms from Q to Q given by $\Delta(r)(q) = rq$ for $r \in R$ and $q \in Q$. Recall from [5, Definition 3.14] that a two-sided ideal I of R is ψ-*compatible* if $I \subseteq \Delta^{-1}(\mathcal{F}_P(Q))$, and *faithful* if $I \cap \ker \Delta = \{0\}$. Throughout the paper, J will denote a fixed faithful and ψ-compatible ideal in R. A covariant representation (S', T', σ', B) of the R-system (P, Q, ψ) is said to be *Cuntz–Pimsner invariant relative to* J if $\pi_{T',S'}(\Delta(x)) = \sigma'(x)$ for all $x \in J$.

We let $\mathcal{O}_{(P,Q,\psi)}(J)$ denote the *Cuntz–Pimsner ring relative to the ideal* J introduced in [5, Definition 3.16]. It follows from [5, Theorem 3.18] that there is a covariant representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$ of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J and universal in the sense that every covariant representation of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J factors through $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$. To ease notation we will let σ, S, T and π denote the maps ι_R^J, ι_P^J, ι_Q^J and π^J, respectively.

As in [5], we recursively define R-bimodules $Q^{\otimes n}$ and $P^{\otimes n}$ for $n \in \mathbb{N}$ by letting $Q^{\otimes 1} = Q$ and $P^{\otimes 1} = P$, and by letting $Q^{\otimes n} = Q^{\otimes n-1} \otimes Q$ and $P^{\otimes n} = P \otimes P^{\otimes n-1}$ for $n > 1$. Let $\psi_1 = \psi$, and define recursively R-bimodule homomorphism $\psi_n : P^{\otimes n} \otimes Q^{\otimes n} \to R$ for $n > 1$ by setting

$$\psi_n\big((p_1 \otimes p_2) \otimes (q_1 \otimes q_2)\big) = \psi\big(p_1 \psi_{n-1}(p_2 \otimes q_1) \otimes q_2\big)$$

for $p_1 \in P$, $p_2 \in P^{\otimes n-1}$, $q_1 \in Q^{\otimes n-1}$ and $q_2 \in Q$. Then $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$ is an R-system for each $n \in \mathbb{N}$. It follows from [5, Lemma 3.8] that since (P, Q, ψ) satisfies condition (FS), so does $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$.

According to [5, Lemma 1.5] there exist for each $n \in \mathbb{N}$ uniquely determined additive maps $T^n : Q^{\otimes n} \to \mathcal{O}_{(P,Q,\psi)}(J)$ and $S^n : P^{\otimes n} \to \mathcal{O}_{(P,Q,\psi)}(J)$ satisfying $T^n(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = T(q_1)T(q_2)\dots T(q_n)$ and $S^n(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = S(p_1)S(p_2)\dots S(p_n)$ for $q_1, q_2, \dots, q_n \in Q$ and $p_1, p_2, \dots, p_n \in P$. It furthermore follows from [5, Lemma 1.5] that the quadruple $(S^n, T^n, \sigma, \mathcal{O}_{(P,Q,\psi)}(J))$ is a covariant representation of the R-system $(P^{\otimes n}, Q^{\otimes}, \psi_n)$. It follows from [5, Theorem 1.9, Proposition 3.1 and Theorem 3.18] that if we for $n \in \mathbb{N}$ let

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(n)} = \operatorname{span}\big(\big\{T^{k+n}(q)S^k(p) \bigm| k \in \mathbb{N},\ q \in Q^{\otimes k+n},\ p \in P^{\otimes k}\big\} \cup \big\{T^n(q) \bigm| q \in Q^{\otimes n}\big\}\big)$$

and

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(-n)} = \operatorname{span}\big(\big\{T^{k}(q)S^{k+n}(p) \bigm| k \in \mathbb{N},\ q \in Q^{\otimes k},\ p \in P^{\otimes k+n}\big\} \cup \big\{S^n(p) \bigm| p \in P^{\otimes n}\big\}\big),$$

and we let

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \operatorname{span}\big(\big\{T^{k}(q)S^{k}(p) \bigm| k \in \mathbb{N},\ q \in Q^{\otimes k},\ p \in P^{\otimes k}\big\} \cup \big\{\sigma(r) \bigm| r \in R\big\}\big),$$

then $(\mathcal{O}_{(P,Q,\psi)}(J)^{(n)})_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading of $\mathcal{O}_{(P,Q,\psi)}(J)$. Recall from [5, Definition 3.20] that an ideal H of $\mathcal{O}_{(P,Q,\psi)}(J)$ is *graded* if $(H \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(n)})_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading of H. When H is a graded ideal, then we let $H^{(n)} = H \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(n)}$ for $n \in \mathbb{Z}$ and write $\bigoplus_{n\in\mathbb{Z}} H^{(n)}$ for H.

We will now recall an example of a relative Cuntz–Pimsner ring given in [5]. This example was our motivating example for developing the results of this paper, and we will return to it throughout the paper to illustrate the results.

Example 2.1. Let $E = (E^0, E^1, r, s)$ be a directed graph, i.e., E^0 and E^1 are sets and r and s are maps from E^1 to E^0, and let F be a field. When n is a positive integer, then we let E^n be the set $\{(e_1, e_2, \dots, e_n) \in E^1 \times E^1 \times \cdots \times E^1 \mid r(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \dots, n-1\}$. For $\alpha = (e_1, e_2, \dots, e_n) \in E^n$ we define $s(\alpha)$ to be $s(e_1)$ and $r(\alpha)$ to be $r(e_n)$. For each $v \in E^0$ we let vE^n denote the set $\{\alpha \in E^n \mid s(\alpha) = v\}$ and we let $E^n v$ denote the set $\{\alpha \in E^n \mid r(\alpha) = v\}$.

Following [5, Example 1.10], we define R_E to be the ring $\bigoplus_{v\in E^0} R_v$ where each R_v is a copy of F. We then define Q_E to be the R_E-bimodule $\bigoplus_{e\in E^1} Q_e$ where each Q_e is a copy of F and the left and the right multiplications are defined by

$$\Big(\sum_{e\in E^1} q_e \mathbf{1}_e\Big) \cdot \Big(\sum_{v\in E^0} r_v \mathbf{1}_v\Big) = \sum_{e\in E^1} q_e r_{r(e)} \mathbf{1}_e,$$

$$\Big(\sum_{v\in E^0} r_v \mathbf{1}_v\Big) \cdot \Big(\sum_{e\in E^1} q_e \mathbf{1}_e\Big) = \sum_{e\in E^1} r_{s(e)} q_e \mathbf{1}_e$$

where $\mathbf{1}_v$ denotes the unit of R_v, $\mathbf{1}_e$ denotes the unit of Q_e, and $\{r_v\}_{v\in E^0}$ and $\{q_e\}_{e\in E^1}$ are families of elements from F with only a finite number of non-zero elements; and we define P_E to be the R_E-bimodule $\bigoplus_{e\in E^1} P_e$ where each P_e is a copy of F and the left and the right multiplications are defined by

$$\left(\sum_{e\in E^1} p_e \mathbf{1}_{\bar{e}}\right)\cdot\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right) = \sum_{e\in E^1} p_e r_{s(e)} \mathbf{1}_{\bar{e}},$$

$$\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right)\cdot\left(\sum_{e\in E^1} p_e \mathbf{1}_{\bar{e}}\right) = \sum_{e\in E^1} r_{r(e)} p_e \mathbf{1}_{\bar{e}}$$

where $\mathbf{1}_{\bar{e}}$ denotes the unit of P_e, and $\{r_v\}_{v\in E^0}$ and $\{p_e\}_{e\in E^1}$ are families of elements from F with only a finite number of non-zero elements. Finally we define $\psi_E : P_E \otimes_{R_E} Q_E \to R_E$ to be the R_E-bimodule homomorphism given by

$$\left(\sum_{e\in E^1} p_e \mathbf{1}_{\bar{e}}\right)\otimes\left(\sum_{e\in E^1} q_e \mathbf{1}_e\right) \mapsto \sum_{v\in E^0}\left(\sum_{e\in E^1 v} p_e q_e\right)\mathbf{1}_v.$$

Then (P_E, Q_E, ψ_E) is an R_E-system which according to [5, Example 5.8] satisfies condition (FS).

Let I be an ideal of R_E and let H be the set $\{v \in E^0 \mid \mathbf{1}_v \in I\}$. Then I is ψ_E-invariant if and only if H is hereditary (that is, whenever $e \in E^1$ with $s(e) \in H$, then $r(e) \in H$).

We can, and will, identify $Q_E^{\otimes n}$ with the R_E-bimodule $\bigoplus_{\alpha\in E^n} Q_\alpha$ where each Q_α is a copy of F and the left and the right multiplications are defined by

$$\left(\sum_{\alpha\in E^n} q_\alpha \mathbf{1}_\alpha\right)\cdot\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right) = \sum_{\alpha\in E^n} q_\alpha r_{r(\alpha)} \mathbf{1}_\alpha,$$

$$\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right)\cdot\left(\sum_{\alpha\in E^n} q_\alpha \mathbf{1}_\alpha\right) = \sum_{\alpha\in E^n} r_{s(\alpha)} q_\alpha \mathbf{1}_\alpha$$

where $\mathbf{1}_\alpha$ denotes the unit of Q_α, and $\{r_v\}_{v\in E^0}$ and $\{q_\alpha\}_{\alpha\in E^n}$ are families of elements of F with only a finite number of non-zero elements. Likewise, we identify $P_E^{\otimes n}$ with the R_E-bimodule $\bigoplus_{\alpha\in E^n} P_\alpha$ where each P_α is a copy of F and the left and the right multiplications are defined by

$$\left(\sum_{\alpha\in E^n} p_\alpha \mathbf{1}_{\bar{\alpha}}\right)\cdot\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right) = \sum_{\alpha\in E^n} p_\alpha r_{r(\alpha)} \mathbf{1}_{\bar{\alpha}},$$

$$\left(\sum_{v\in E^0} r_v \mathbf{1}_v\right)\cdot\left(\sum_{\alpha\in E^n} p_\alpha \mathbf{1}_{\bar{\alpha}}\right) = \sum_{\alpha\in E^n} r_{s(\alpha)} p_\alpha \mathbf{1}_{\bar{\alpha}}$$

where $\mathbf{1}_{\bar{\alpha}}$ denotes the unit of P_α, and $\{r_v\}_{v\in E^0}$ and $\{p_\alpha\}_{\alpha\in E^n}$ are families of elements of F with only a finite number of non-zero elements. We then have that $(\psi_E)_n : P_E^{\otimes n} \otimes Q_E^{\otimes n} \to R_E$ is given by

$$\left(\sum_{\alpha\in E^n} p_\alpha \mathbf{1}_{\bar{\alpha}}\right)\otimes\left(\sum_{\alpha\in E^n} q_\alpha \mathbf{1}_\alpha\right) \mapsto \sum_{v\in E^0}\left(\sum_{\alpha\in E^n v} p_\alpha q_\alpha\right)\mathbf{1}_v.$$

Let Δ be the ring homomorphism from R_E to the ring of right R_E-module homomorphisms from Q_E to Q_E given by $\Delta(r)(q) = rq$ for $r \in R_E$ and $q \in Q_E$. It is shown in [5, Example 5.8] that $\ker\Delta = \operatorname{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } vE^1 = \emptyset\}$ and $\Delta^{-1}(\mathcal{F}_{P_E}(Q_E)) = \operatorname{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } vE^1 \text{ is finite}\}$. It follows that if we let J_E be the ideal $\operatorname{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } 0 < |vE^1| < \infty\} \subseteq R_E$, then J_E is a faithful, ψ-compatible ideal.

Let $(\iota_{P_E}^{J_E}, \iota_{Q_E}^{J_E}, \iota_{R_E}^{J_E}, \mathcal{O}_{(P_E,Q_E,\psi)}(J_E))$ be the covariant representation of (P, Q, ψ) given in [5, Definition 3.16]. It is shown in [5, Example 5.8] that if we for each $v \in E^0$ let $p_v = \iota_{P_E}^{J_E}(\mathbf{1}_v)$, and for each

$e \in E^1$ let $x_e = \iota_{Q_E}^{J_E}(\mathbf{1}_e)$ and $y_e = \iota_{P_E}^{J_E}(\mathbf{1}_{\bar{e}})$, then $\mathcal{O}_{(P_E,Q_E,\psi)}(J_E)$ is generated by $\{p_v \mid v \in E^0\} \cup \{x_e \mid e \in E^1\} \cup \{y_e \mid e \in E^1\}$ and these elements satisfy

(i) $p_{s(e)}x_e = x_e = x_e p_{r(e)}$ for $e \in E^1$,
(ii) $p_{r(e)}y_e = y_e = y_e p_{s(e)}$ for $e \in E^1$,
(iii) $y_e x_f = \delta_{e,f} p_{r(e)}$ for $e, f \in E^1$,
(iv) $p_v = \sum_{e\in vE^1} x_e y_e$ for $v \in E^0$ with $0 < |vE^1| < \infty$.

In fact, it is shown that $\mathcal{O}_{(P_E,Q_E,\psi)}(J_E)$ is isomorphic to the Leavitt path algebra $L_F(E)$ of E (see for example [1,2,19]).

3. The ideal intersection property

This section contains some preliminary results leading to Proposition 3.8, which is pivotal for the rest of the paper. We recall that R denotes a fixed (associative, but not necessarily unital) ring, (P, Q, ψ) a fixed R-system satisfying condition (FS), J a fixed faithful and ψ-compatible ideal in R, and that $(S, T, \sigma, \mathcal{O}_{(P,Q,\psi)}(J))$ denotes the universal covariant representation which is Cuntz–Pimsner invariant relative to the ideal J.

Lemma 3.1. *If $n \in \mathbb{N}$, $x_{-n} \in \mathcal{O}_{(P,Q,\psi)}(J)^{(-n)} \setminus \{0\}$ and $x_n \in \mathcal{O}_{(P,Q,\psi)}(J)^{(n)} \setminus \{0\}$, then there are $p \in P^{\otimes n}$ and $q \in Q^{\otimes n}$ such that $x_{-n}T^n(q) \neq 0$ and $S^n(p)x_n \neq 0$.*

Proof. Write x_n as $\sum_{i=1}^k T^n(q_i)y_i$ where $q_i \in Q^{\otimes n}$ and $y_i \in \mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ for $i = 1, 2, \dots, k$. It follows from condition (FS) that there is a $\Theta \in \mathcal{F}_{P^{\otimes n}}(Q^{\otimes n})$ such that $\Theta(q_i) = q_i$ for each $i = 1, 2, \dots, k$. It follows that $S^n(p)x_n$ cannot be 0 for all $p \in P^{\otimes n}$. That $x_{-n}T^n(q) \neq 0$ for some $q \in Q^{\otimes n}$ can be proved in a similar way. $\square$

Definition 3.2. For an ideal I in R, let $\psi^{-1}(I)$ be the ideal

$$\{x \in R \mid \psi(px \otimes q) \in I \text{ for all } q \in Q \text{ and all } p \in P\},$$

and let $I^{[\infty]}$ be the ideal

$$\bigcap_{k=1}^{\infty} I^{[k]}$$

where $I^{[k]}$ is defined recursively by $I^{[1]} = I$ and $I^{[k]} = \psi^{-1}(I^{[k-1]}) \cap I$ for $k > 1$.

Notice that I is ψ-invariant if and only if $I \subseteq \psi^{-1}(I)$.

Example 3.3. Let E be a directed graph, F a field, and let R_E and (P_E, Q_E, ψ_E) be as in Example 2.1. Let I be an ideal of R_E and let $H = \{v \in E^0 \mid \mathbf{1}_v \in I\}$. Then $I = \operatorname{span}_F\{\mathbf{1}_v \mid v \in H\}$ and

$$\psi_E^{-1}(I) = \operatorname{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } r(e) \in H \text{ for all } e \in vE^1\}.$$

It follows that

$$I^{[k]} = \operatorname{span}_F\left\{\mathbf{1}_v \;\middle|\; v \in H \text{ and } r(e) \in H \text{ for all } e \in \bigcup_{i=1}^{k-1} vE^i\right\}$$

for $k > 1$, and

$$I^{[\infty]} = \operatorname{span}_F \left\{ \mathbf{1}_v \;\middle|\; v \in H \text{ and } r(e) \in H \text{ for all } e \in \bigcup_{i=1}^{\infty} vE^i \right\}.$$

Recall from [5, Definition 7.1] that if I is an ideal in R, then $QI = \operatorname{span}\{qx \mid q \in Q,\ x \in I\}$.

Lemma 3.4. *Let $x \in R$. Then $x \in \psi^{-1}(I)$ if and only if $xq \in QI$ for all $q \in Q$.*

Proof. Assume first that $x \in \psi^{-1}(I)$ and that $q \in Q$. Then it follows from condition (FS) that there are $q_1, \dots, q_m \in Q$ and $p_1, \dots, p_m \in P$ such that $xq = \sum_{i=1}^m q_i \psi(p_i \otimes xq)$. Since each $\psi(p_i \otimes xq) \in I$, it follows that $xq \in QI$.

Assume then that $x \in R$ and $xq \in QI$ for all $q \in Q$, and let $q \in Q$ and $p \in P$. Then there are $q_1, \dots, q_m \in Q$ and $x_1, \dots, x_m \in I$ such that $xq = \sum_{i=1}^m q_i x_i$, from which it follows that $\psi(px \otimes q) = \psi(p \otimes xq) = \sum_{i=1}^m \psi(p \otimes q_i) x_i \in I$. Thus $x \in \psi^{-1}(I)$. $\square$

Let us now specialize to the case where $I = J$.

Example 3.5. Let E be a directed graph, F a field, and let R_E, (P_E, Q_E, ψ_E) and J_E be as in Example 2.1. It follows from Example 3.3 that

$$J_E^{[k]} = \operatorname{span}_F \left\{ \mathbf{1}_v \;\middle|\; v \in E^0 \text{ and } 0 < \left| vE^i \right| < \infty \text{ for } i = 1, 2, \dots, k \right\}$$

for each $k \in \mathbb{N}$, and that

$$J_E^{[\infty]} = \operatorname{span}_F \left\{ \mathbf{1}_v \;\middle|\; v \in E^0 \text{ and } 0 < \left| vE^i \right| < \infty \text{ for all } i \in \mathbb{N} \right\}.$$

Lemma 3.6. *Let $k \in \mathbb{N}$ and $x \in R$. Then $x \in J^{[k]}$ if and only if $\sigma(x) \in \operatorname{span}\{T^k(q)S^k(p) \mid q \in Q^{\otimes k},\ p \in P^{\otimes k}\}$.*

Proof. We will prove the lemma by induction over k. For $k = 1$ the lemma follows from [5, Proposition 3.28].

Assume now that $k > 1$ and that $x \in J^{[k-1]}$ if and only if $\sigma(x) \in \operatorname{span}\{T^{k-1}(q)S^{k-1}(p) \mid q \in Q^{\otimes k-1},\ p \in P^{\otimes k-1}\}$. Let $x \in R$. We will then prove that $x \in J^{[k]}$ if and only if $\sigma(x) \in \operatorname{span}\{T^k(q)S^k(p) \mid q \in Q^{\otimes k},\ p \in P^{\otimes k}\}$. If $x \in J^{[k]} = \psi^{-1}(J^{[k-1]}) \cap J$, then it follows from [5, Proposition 3.28] that there are $q_1, \dots, q_m \in Q$ and $p_1, \dots, p_m \in P$ such that $\sigma(x) = \sum_{i=1}^m T(q_i)S(p_i)$. It follows from condition (FS) that there are $q'_1, \dots, q'_n \in Q$ and $p'_1, \dots, p'_n \in P$ such that $\sum_{j=1}^n \theta_{p'_j, q'_j}(p_i) = p_i$ for each i, from which it follows that

$$\sigma(x) = \sum_{i=1}^m T(q_i)S(p_i) = \sum_{i=1}^m \sum_{j=1}^n T(q_i)S(p_i)T\left(q'_j\right)S\left(p'_j\right) = \sum_{j=1}^n T\left(xq'_j\right)S\left(p'_j\right).$$

It follows from Lemma 3.4 that there for each j are $q_{j,1} \dots, q_{j,m_j} \in Q$ and $x_{j,1}, \dots, x_{j,m_j} \in J^{[k-1]}$ such that $xq_j = \sum_{l=1}^{m_j} q_{j,l} x_{j,l}$, and it then follows from the induction hypothesis that

$$\begin{aligned}
\sigma(x) &= \sum_{j=1}^n T\left(xq'_j\right)S\left(p'_j\right) \\
&= \sum_{j=1}^n \sum_{l=1}^{m_j} T(q_{j,l})\sigma(x_{j,l})S\left(p'_j\right) \in \operatorname{span}\left\{T^k(q)S^k(p) \;\middle|\; q \in Q^{\otimes k},\ p \in P^{\otimes k}\right\}.
\end{aligned}$$

Conversely, if $\sigma(x) = \sum_{i=1}^m T^k(q_i)S^k(p_i)$, then $\iota_R(x) - \sum_{i=1}^m \iota_Q^k(q_i)\iota_P^k(p_i) \in \mathcal{T}(J)$ (cf. [5, Definitions 3.15 and 3.16]), so it follows from [5, Lemma 3.21] that $x \in J$. If $p \in P$ and $q \in Q$, then

$$\sigma\big(\psi(px \otimes q)\big) = S(p)\sum_{i=1}^m T^k(q_i)S^k(p_i)T(q) \in \operatorname{span}\big\{T^{k-1}\big(q'\big)S^{k-1}\big(p'\big) \bigm| q' \in Q^{\otimes k-1},\ p' \in P^{\otimes k-1}\big\},$$

which together with the induction hypothesis implies that $\psi(px \otimes q) \in J^{[k-1]}$, and thus that $x \in \psi^{-1}(J^{[k-1]}) \cap J = J^{[k]}$. □

The following property was studied, for example, in the context of strongly G-graded rings in [13].

Definition 3.7. A subring A of $\mathcal{O}_{(P,Q,\psi)}(J)$ has the *ideal intersection property* if the implication $K \cap A = \{0\} \Rightarrow K = \{0\}$ holds for every ideal K in $\mathcal{O}_{(P,Q,\psi)}(J)$.

We of course have that $\mathcal{O}_{(P,Q,\psi)}(J)$ itself has the ideal intersection property. We will in this paper study when $\sigma(R)$ and $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ have the ideal intersection property. We begin with $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$.

Let $n \in \mathbb{N}$. Recall from [5, Section 2] that there for each $p \in P$ exists a unique R-bimodule homomorphism $S_p : Q^{\otimes n+1} \to Q^{\otimes n}$ characterized by $S_p(q \otimes q_n) = \psi(p \otimes q)q_n$ for $q \in Q$ and $q_n \in Q^{\otimes n}$. Similarly, there exists for each $q_n \in Q^{\otimes n}$ an R-bimodule homomorphism $T_{q_n} : Q \to Q^{\otimes n+1}$ given by $T_{q_n}(q) = q_n \otimes q$ for $q \in Q$. Notice that $T^n(S_p(T_{q_n}(q))) = S(p)T^n(q_n)T(q)$ for $p \in P$, $q_n \in Q^{\otimes n}$ and $q \in Q$.

Proposition 3.8. *The following* 3 *conditions are equivalent*:

(1) *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ does not have the ideal intersection property.*
(2) *There is a non-zero graded ideal $\bigoplus_{k\in\mathbb{Z}} H^{(k)}$ in $\mathcal{O}_{(P,Q,\psi)}(J)$, an $n \in \mathbb{N}$ and a family $(\phi_k)_{k\in\mathbb{Z}}$ of injective $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$-bimodule homomorphisms $\phi_k : H^{(k)} \to \mathcal{O}_{(P,Q,\psi)}(J)^{(k+n)}$ such that $x\phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ for $k, j \in \mathbb{Z}$, $x \in \mathcal{O}_{(P,Q,\psi)}(J)^{(j)}$ and $y \in H^{(k)}$.*
(3) *There is a non-zero ψ-invariant ideal I_0 of R, an $n \in \mathbb{N}$ and an injective R-bimodule homomorphism $\eta : I_0 \to Q^{\otimes n}$ such that $S_pT_{\eta(x)}(q) = \eta(\psi(px\otimes q))$ for $p \in P$, $x \in I_0$ and $q \in Q$, and such that $I_0 \subseteq J^{[\infty]}$.*

Proof. First we will prove that (1) $\Rightarrow$ (2): Let K be a non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ such that $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$. Let N be the set of $n \in \mathbb{N}_0$ for which there are $x_i \in \mathcal{O}_{(P,Q,\psi)}(J)^{(i)}$, $i = 0, 1, \dots, n$ with $x_0 \neq 0$ such that $\sum_{i=0}^n x_i \in K$. Let $\sum_{i=j}^k x_i \in K$ where $j \leqslant k \in \mathbb{Z}$, $x_i \in \mathcal{O}_{(P,Q,\psi)}(J)^{(i)}$ for $i = j, j+1, \dots, k$ and $x_j \neq 0$. If $j \neq 0$, then it follows from Lemma 3.1 that there is a $y_{-j} \in \mathcal{O}_{(P,Q,\psi)}(J)^{(-j)}$ such that either $y_{-j}x_j$ or x_jy_{-j} is non-zero. It follows that $N \neq \emptyset$. Since $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$, it follows that $0 \notin N$. Let $n = \min N$. Then $n \in \mathbb{N}$.

For each $k \in \mathbb{Z}$ let

$$H^{(k)} := \left\{x_k \in \mathcal{O}_{(P,Q,\psi)}(J)^{(k)} \;\middle|\; \exists x_{k+i} \in \mathcal{O}_{(P,Q,\psi)}(J)^{(k+i)},\ i = 1, 2, \dots, n\colon\ \sum_{i=0}^n x_{k+i} \in K\right\}.$$

If $x \in H^{(k)}$ and $y \in \mathcal{O}_{(P,Q,\psi)}(J)^{(j)}$, then $xy, yx \in H^{(k+j)}$. It follows that $\bigoplus_{k\in\mathbb{Z}} H^{(k)}$ is a graded ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$, and since $H^{(0)} \neq \{0\}$, it must be the case that $\bigoplus_{k\in\mathbb{Z}} H^{(k)}$ is non-zero.

Let $k \in \mathbb{Z}$ and let $x_k \in H^{(k)}$. It follows from Lemma 3.1 and the minimality of n that there is a unique $x_{k+n} \in \mathcal{O}_{(P,Q,\psi)}(J)^{(k+n)}$ satisfying that there exist $x_{k+i} \in \mathcal{O}_{(P,Q,\psi)}(J)^{(k+i)}$, $i = 1, 2, \dots, n-1$ such that $\sum_{i=0}^n x_{k+i} \in K$. It also follows from Lemma 3.1 and the minimality of n that $x_{k+n} \neq 0$ if $x_k \neq 0$. Thus there is an injective map $\phi_k : H^{(k)} \to \mathcal{O}_{(P,Q,\psi)}(J)^{(k+n)}$ sending x_k to x_{k+n}. It is easy to check that ϕ_k is an $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$-bimodule homomorphism, and that $x\phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ when $k, j \in \mathbb{Z}$, $x \in \mathcal{O}_{(P,Q,\psi)}(J)^{(j)}$ and $y \in H^{(k)}$. Thus (2) is proved.

Next let us check that (2) $\Rightarrow$ (3): We will first prove that $H^{(0)} \cap \sigma(R) \neq \{0\}$, so assume, for contradiction, that $H^{(0)} \cap \sigma(R) = \{0\}$. Then it follows from [5, Lemma 3.21 and Theorem 7.27] that

$$\begin{aligned} H^{(0)} = \operatorname{span}\big(&\big\{T^n(q)\big(\sigma(x) - \pi\big(\Delta(x)\big)\big)S^n(p) \bigm| n \in \mathbb{N},\ q \in Q^{\otimes n},\ x \in J',\ p \in P^{\otimes n}\big\} \\ &\cup \big\{\sigma(x) - \pi\big(\Delta(x)\big) \bigm| x \in J'\big\}\big) \end{aligned}$$

for some faithful ψ-compatible ideal J' of R which contains J. We claim that $H^{(0)}$ must contain a non-zero element of the form $\sigma(x) - \pi(\Delta(x))$, $x \in J'$. To see that this is the case, let y be a non-zero element of $H^{(0)}$ and write it as

$$\sigma(x_0) - \pi\big(\Delta(x_0)\big) + \sum_{i=1}^{k} T^{n_i}(q_i)\big(\sigma(x_i) - \pi\big(\Delta(x_i)\big)\big)S^{n_i}(p_i)$$

where $k \in \mathbb{N}$, $x_0, x_1, \ldots, x_k \in J'$ and $n_i \in \mathbb{N}$, $q_i \in Q^{\otimes n_i}$, $p_i \in P^{\otimes n_i}$ for each $i \in \{1, 2, \ldots, k\}$, and assume that $\sum_{i\in M} T^{n_i}(q_i)(\sigma(x_i) - \pi(\Delta(x_i)))S^{n_i}(p_i) \neq 0$ where M is the set of those i's for which n_i is maximal among $\{n_1, n_2, \ldots, n_k\}$. Let n be the maximal value of n_i. It follows from condition (FS) that there are $q \in Q^{\otimes n}$ and $p \in P^{\otimes n}$ such that if we let $x = \sum_{i\in M} \psi_n(p \otimes q_i)x_i\psi_n(p_i \otimes q)$, then

$$\sigma(x) - \pi\big(\Delta(x)\big) = S^n(p)\sum_{i\in M} T^{n_i}(q_i)\big(\sigma(x_i) - \pi\big(\Delta(x_i)\big)\big)S^{n_i}(p_i)T^n(q) \neq 0.$$

Since $(\sigma(x_0) - \pi(\Delta(x_0)))T^n(q) = 0$ and $(\sigma(x_i) - \pi(\Delta(x_i)))S^{n_i}(p_i)T^n(q) = 0$ for each $i \notin M$, it follows that

$$\begin{aligned} &\sigma(x) - \pi\big(\Delta(x)\big) \\ &\quad = S^n(p)\left(\sigma(x_0) - \pi\big(\Delta(x_0)\big) + \sum_{i=1}^{k} T^{n_i}(q_i)\big(\sigma(x_i) - \pi\big(\Delta(x_i)\big)\big)S^{n_i}(p_i)\right)T^n(q) \in H^{(0)}. \end{aligned}$$

Thus $H^{(0)}$ contains a non-zero element of the form $\sigma(x) - \pi(\Delta(x))$, $x \in J'$. If follows from condition (FS) that there is a $p' \in P^{\otimes n}$ such that

$$S^n\big(p'\big)\phi_0\big(\sigma(x) - \pi\big(\Delta(x)\big)\big) \neq 0,$$

but since $S^n(p'')(\sigma(x) - \pi(\Delta(x))) = 0$ for all $p'' \in P^{\otimes n}$, it follows that

$$S^n\big(p'\big)\phi_0\big(\sigma(x) - \pi\big(\Delta(x)\big)\big) = \phi_{-n}\big(S^n\big(p'\big)\big(\sigma(x) - \pi\big(\Delta(x)\big)\big)\big) = 0,$$

and we have reached a contradiction. Thus it must be the case that $H^{(0)} \cap \sigma(R) \neq \{0\}$. Let $I = \{x \in R \mid \sigma(x) \in H^{(0)}\}$. Then I is a non-zero ψ-invariant ideal of R. For each $m \in \mathbb{N}_0$ let

$$A_m = \operatorname{span}\big\{T^{n+k}(q)S^k(p) \bigm| k \in \{0, 1, \ldots, m\},\ q \in Q^{\otimes n+k},\ p \in P^{\otimes k}\big\} \subseteq \mathcal{O}_{(P,Q,\psi)}(J)^{(n)}$$

and

$$I_m = \big\{x \in I \bigm| \phi_0\big(\sigma(x)\big) \in A_m\big\}.$$

Then $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ and each I_m is a ψ-invariant two-sided ideal in R. In fact, $x \in I_{m+1}$, implies that $\psi(px \otimes q) \in I_m$ for all $p \in P$ and $q \in Q$. Since I is non-zero, there exists an $x \neq 0$ and an $m \in \mathbb{N}_0$ such that $x \in I_m$. Choose $k \in \mathbb{N}$ such that $kn \geqslant m$. Then

$$\phi_{(k-1)n} \circ \phi_{(k-2)n} \circ \cdots \circ \phi_n \circ \phi_0\big(\sigma(x)\big) \in \mathcal{O}_{(P,Q,\psi)}(J)^{(nk)} \setminus \{0\}$$

so it follows from Lemma 3.1 that there is a $p \in P^{\otimes nk}$ such that

$$\phi_{-n} \circ \phi_{-2n} \circ \cdots \circ \phi_{-(k-1)n} \circ \phi_{-kn}\big(S^{nk}(px)\big) = S^{nk}(p)\phi_{(k-1)n} \circ \phi_{(k-2)n} \circ \cdots \circ \phi_n \circ \phi_0\big(\sigma(x)\big) \neq 0,$$

from which it follows that $px \neq 0$. It follows from condition (FS) that there is a $q \in Q^{\otimes kn}$ such that $\psi_{kn}(px \otimes q) \neq 0$. We have that $\psi_{kn}(px \otimes q) \in I_0$, so $I_0 \neq \{0\}$.

Since $\phi_0(\sigma(x)) \in T^n(Q^{\otimes n})$ for every $x \in I_0$, and $T^n : Q^{\otimes n} \to \mathcal{O}_{(P,Q,\psi)}(J)^{(n)}$ is injective, we can define $\eta : I_0 \to Q^{\otimes n}$ by, for $x \in I_0$, letting $\eta(x)$ be the unique element of $Q^{\otimes n}$ such that $T^n(\eta(x)) = \phi_0(\sigma(x))$. It is straightforward to check that η is an injective R-bimodule homomorphism, and if $p \in P$, $x \in I_0$ and $q \in Q$, then

$$\begin{aligned} T^n\big(\eta\big(\psi(px \otimes q)\big)\big) &= \phi_0\big(\sigma\big(\psi(px \otimes q)\big)\big) = S(p)\phi_0\big(\sigma(x)\big)T(q) \\ &= S(p)T^n\big(\eta(x)\big)T(q) = T^n\big(S_p T_{\eta(x)}(q)\big), \end{aligned}$$

from which it follows that $\eta(\psi(px \otimes q)) = S_p T_{\eta(x)}(q)$.

If $x \in I_0$ then it follows from condition (FS) that there are $q_i \in Q^{(n)}$, $p_i \in P^{(n)}$, $i = 1, 2, \ldots, m$ such that $\sum_{i=0}^m \theta_{q_i,p_i}\eta(x) = \eta(x)$. We then have that

$$T^n\big(\eta(x)\big) = T^n\left(\sum_{i=0}^m \theta_{q_i,p_i}\eta(x)\right) = \sum_{i=0}^m T^n(q_i)S^n(p_i)\phi_0\big(\sigma(x)\big) = \phi_0\left(\sum_{i=0}^m T^n(q_i)S^n(p_i x)\right)$$

from which it follows that $\sigma(x) = \sum_{i=0}^m T^n(q_i)S^n(p_i x)$. It now follows from Lemma 3.6 that $x \in J^{[n]} \subseteq J$. Since I_0 is ψ-invariant, it follows that $x \in J^{[\infty]}$. Hence (3) holds.

Finally let us prove that (3) $\Rightarrow$ (1): Let K be the ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ generated by $\{\sigma(x) - T^n(\eta(x)) \mid x \in I_0\}$. Clearly, K is non-zero, so we just have to prove that $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$. Using condition (FS) and the properties of η, one can show that if $p \in P$, $x \in I_0$ and $q \in Q$, then

$$S(p)\big(\sigma(x) - T^n\big(\eta(x)\big)\big) \in \operatorname{span}\big\{\big(\sigma(x') - T^n\big(\eta(x')\big)\big)S(p') \bigm| x' \in I_0,\ p' \in P\big\}$$

and

$$\big(\sigma(x) - T^n\big(\eta(x)\big)\big)T(q) \in \operatorname{span}\big\{T(q')\big(\sigma(x') - T^n\big(\eta(x')\big)\big) \bigm| q' \in Q,\ x' \in I_0\big\}.$$

It follows that

$$\begin{aligned} K = \operatorname{span}\big(&\big\{T^k(q)\big(\sigma(x) - T^n\big(\eta(x)\big)\big) \bigm| k \in \mathbb{N},\ q \in Q^{\otimes k},\ x \in I_0\big\} \\ &\cup \big\{T^k(q)\big(\sigma(x) - T^n\big(\eta(x)\big)\big)S^l(p) \bigm| k, l \in \mathbb{N},\ q \in Q^{\otimes k},\ x \in I_0,\ p \in P^{\otimes l}\big\} \\ &\cup \big\{\sigma(x) - T^n\big(\eta(x)\big) \bigm| x \in I_0\big\} \\ &\cup \big\{T^k(q)\big(\sigma(x) - T^n\big(\eta(x)\big)\big) \bigm| l \in \mathbb{N},\ x \in I_0,\ p \in P^{\otimes l}\big\}\big), \end{aligned}$$

so to show that $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$, it suffices to show the following 3 things:

(i) if $l \in \mathbb{N}$, A is a finite subset of $\{(k,q,x,p) \mid k \in \mathbb{N},\ q \in Q^{\otimes l+k},\ x \in I_0,\ p \in P^{\otimes k}\}$ and B is a finite subset of $\{(q,x) \mid q \in Q^{\otimes l},\ x \in I_0\}$, then

$$\sum_{(k,q,x,p)\in A} T^{l+k}(q)\sigma(x)S^k(p) + \sum_{(q,x)\in B} T^l(q)\sigma(x) = 0$$

if and only if

$$\sum_{(k,q,x,p)\in A} T^{l+k}(q)T^n\big(\eta(x)\big)S^k(p) + \sum_{(q,x)\in B} T^l(q)T^n\big(\eta(x)\big) = 0,$$

(ii) if A is a finite subset of $\{(k,q,x,p) \mid k \in \mathbb{N},\ q \in Q^{\otimes k},\ x \in I_0,\ p \in P^{\otimes k}\}$ and $x_0 \in I_0$, then

$$\sum_{(k,q,x,p)\in A} T^k(q)\sigma(x)S^k(p) + \sigma(x_0) = 0$$

if and only if

$$\sum_{(k,q,x,p)\in A} T^k(q)T^n\big(\eta(x)\big)S^k(p) + T^n\big(\eta(x_0)\big) = 0,$$

(iii) if $l \in \mathbb{N}$, A is a finite subset of $\{(k,q,x,p) \mid k \in \mathbb{N},\ q \in Q^{\otimes k},\ x \in I_0,\ p \in P^{\otimes l+k}\}$ and B is a finite subset of $\{(x,p) \mid x \in I_0,\ p \in P^{\otimes l+k}\}$, then

$$\sum_{(k,q,x,p)\in A} T^k(q)\sigma(x)S^{l+k}(p) + \sum_{(q,x)\in B} \sigma(x)S^{l+k}(p) = 0$$

if and only if

$$\sum_{(k,q,x,p)\in A} T^k(q)T^n\big(\eta(x)\big)S^{l+k}(p) + \sum_{(x,p)\in B} T^n\big(\eta(x)\big)S^l(p) = 0.$$

We will just prove (i). The other two claims can be proved in a similar way.

To prove (i), notice first that if $x \in I_0$ and $k \in \mathbb{N}$, then, since $I_0 \subseteq J^{[\infty]} \subseteq J^{[k]}$, it follows from Lemma 3.6 that there are $q_1, \dots, q_m \in Q^{\otimes k}$ and $p_1, \dots, p_m \in P^{\otimes k}$ such that $\sigma(x) = \sum_{i=1}^m T^k(q_i)S^k(p_i)$. It follows from condition (FS) that there are $q'_1, \dots, q'_r, q''_1, \dots, q''_s \in Q^{\otimes k}$ and $p'_1, \dots, p'_r, p''_1, \dots, p''_s \in P^{\otimes k}$ such that

$$\begin{aligned}\sigma(x) &= \sum_{i=1}^m T^k(q_i)S^k(p_i) = \sum_{j=1}^r \sum_{l=1}^s \sum_{i=1}^m T^k\big(q'_j\big)S^k\big(p'_j\big)T^k(q_i)S^k(p_i)T^k\big(q''_l\big)S^k\big(p''_l\big)\\ &= \sum_{j=1}^r \sum_{l=1}^s T^k\big(q'_j\big)S^k\big(p'_j\big)\sigma(x)T^k\big(q''_l\big)S^k\big(p''_l\big) = \sum_{j=1}^r \sum_{l=1}^s T^k\big(q'_j\big)\sigma\big(\psi_k\big(p'_j x \otimes q''_l\big)\big)S^k\big(p''_l\big).\end{aligned}$$

Since I_0 is ψ-invariant, it follows that each $\psi_k(p'_j x \otimes q''_l) \in I_0$ and thus that

$$T^n\big(\eta(x)\big) = \sum_{j=1}^r \sum_{l=1}^s T^k\big(q'_j\big)T^n\big(\eta\big(\psi_k\big(p'_j x \otimes q''_l\big)\big)\big)S^k\big(p''_l\big).$$

Thus it suffices to show that if $k, l \in \mathbb{N}$ and C is a finite subset of $\{(q, x, p) \mid q \in Q^{l+k},\ x \in I_0,\ p \in P^{\otimes k}\}$, then it is the case that $\sum_{(q,x,p)\in C} T^{l+k}(q)\sigma(x)S^k(p) = 0$ if and only if $\sum_{(q,x,p)\in C} T^{l+k}(q)T^n(\eta(x)) \times S^k(p) = 0$, and that can be done using condition (FS) and the properties of η. $\square$

4. Condition (L)

In this section condition (L) is introduced (Definition 4.1) and sufficient and necessary conditions for when every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal (Theorem 4.2) are given.

Definition 4.1. We say that a ψ-invariant ideal I in R is a *ψ-invariant cycle* if there exist $n \in \mathbb{N}$ and an injective R-bimodule homomorphism $\eta : I \to Q^{\otimes n}$ such that $S_p T_{\eta(x)}(q) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I$ and $q \in Q$, and we say that J satisfies *condition* (*L*) with respect to the R-system (P, Q, ψ) if there are no non-zero ψ-invariant cycles I in R such that $I \subseteq J^{[\infty]}$.

We will often, when it is clear from the context which R-system (P, Q, ψ) we are working with, simply call a ψ-invariant cycle for an invariant cycle, and say that J satisfies condition (L) instead of saying that it satisfies condition (L) with respect to (P, Q, ψ).

Recall from [5, Definition 3.23] that if (S', T', σ', B) is a covariant representation of (P, Q, ψ), then $J_{(S',T',\sigma',B)}$ is defined to be the ideal $\{x \in R \mid \sigma'(x) \in \pi_{T',S'}(\mathcal{F}_P(Q))\}$, and recall from [5, Definitions 1.2 and 3.3] that (S', T', σ', B) is said to be *injective* if σ' is injective, *surjective* if B is generated (as a ring) by $\sigma'(R) \cup T'(Q) \cup S'(P)$, and *graded* if there is a $\mathbb{Z}$-grading $(B^{(n)})_{n\in\mathbb{Z}}$ of B such that $\sigma'(R) \subseteq B^{(0)}$, $T'(Q) \subseteq B^{(1)}$ and $S'(P) \subseteq B^{(-1)}$.

Theorem 4.2. *The following 4 conditions are equivalent*:

(1) *The ideal J satisfies condition (L).*
(2) *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property.*
(3) *Every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal.*
(4) *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) and $J = J_{(S',T',\sigma',B)}$, then the ring homomorphism $\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ from [5, Theorem 3.29(ii)] is injective.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 3.8.

(2) $\Rightarrow$ (3): Let K be a non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$. Then $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} \neq \{0\}$ by assumption, and it follows from [5, Lemma 3.35] that the ideal H generated by $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ is graded. Since H is obviously contained in K, this proves (3).

(3) $\Rightarrow$ (2): Let K be a non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$. By assumption there is a non-zero graded ideal H such that $H \subseteq K$. It follows from [5, Lemma 3.35] that $H \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} \neq \{0\}$, so also $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} \neq \{0\}$, which proves that $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property.

(2) $\Rightarrow$ (4): Let H be the ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ generated by $\ker \eta^J_{(S',T',\sigma',B)} \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$, and let $\wp : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{(P,Q,\psi)}(J)/H$ be the quotient map. Then $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/H)$ is a surjective covariant representation of (P, Q, ψ). It follows from [5, Lemma 3.35] that H is graded, from which it follows that the representation $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/H)$ is graded (see [5, Definition 3.20]). Since $H \subseteq \ker \eta^J_{(S',T',\sigma',B)}$, it follows that there is a ring homomorphism $\phi : \mathcal{O}_{(P,Q,\psi)}(J)/H \to B$ such that $\phi \circ \wp = \eta^J_{(S',T',\sigma',B)}$ and $\phi \circ \wp \circ S = S'$, $\phi \circ \wp \circ T = T'$ and $\phi \circ \wp \circ \sigma = \sigma'$. Since (S', T', σ', B) is an injective representation, it follows that also $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/H)$ is injective. It follows from [5, Remark 3.13] that

$$J \subseteq J_{(\wp\circ S, \wp\circ T, \wp\circ\sigma, \mathcal{O}_{(P,Q,\psi)}(J)/H)} \subseteq J_{(S',T',\sigma',B)} = J.$$

Thus $J_{(\wp\circ S,\wp\circ T,\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/H)} = J$, and it follows from [5, Theorem 3.29] that $\wp$ is an isomorphism, and thus that $\ker\eta^J_{(S',T',\sigma',B)} \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$. It follows by assumption that $\ker\eta^J_{(S',T',\sigma',B)} = \{0\}$, and thus that $\eta^J_{(S',T',\sigma',B)}$ is injective.

(4) $\Rightarrow$ (2): Let K be an ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ such that $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$, and let $\wp : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{(P,Q,\psi)}(J)/K$ be the quotient map. Then $(\wp\circ S, \wp\circ T, \wp\circ\sigma, \mathcal{O}_{(P,Q,\psi)}(J)/K)$ is a surjective covariant representation of (P, Q, ψ). Since $\sigma(R)$ and $\pi_{T,S}(\mathcal{F}_P(Q))$ are subsets of $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ and $K \cap \mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \{0\}$, it follows from [5, Proposition 3.28] that

$$J_{(\wp\circ S,\wp\circ T,\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/H)} = J_{(S,T,\sigma,\mathcal{O}_{(P,Q,\psi)}(J))} = J.$$

Thus $\wp = \eta_{(\wp\circ S,\wp\circ T\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/K)}$ is injective by assumption, and $K = \{0\}$ which proves that $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property. $\square$

Example 4.3. Let E be a directed graph, F a field, and let R_E, (P_E, Q_E, ψ_E) and J_E be as in Example 2.1. As in [19, Definition 6.1], we define a *closed path* to be an $\alpha \in E^n$ such that $r(\alpha) = s(\alpha)$. The element $s(\alpha)$ is called the *base* of α. A closed path $\alpha = (e_1, e_2, \dots, e_n)$ is said to be *simple* if $s(e_i) \neq s(e_1)$ for each $i = 2, 3, \dots, n$ (cf. [19, Definition 6.2]), and to have an *exit* if $|s(e_i)E^1| > 1$ for some $i \in \{1, 2, \dots, n\}$.

Suppose that I is a non-zero ψ_E-invariant cycle and let $\eta : I \to Q_E^{\otimes n}$ be an injective R_E-bimodule homomorphism satisfying $S_p T_{\eta(x)}(q) = \eta(\psi_E(px \otimes q))$ for $p \in P_E$, $x \in I$ and $q \in Q_E$. We will prove that it follows that (E^0, E^1, r, s) has a closed path without an exit.

Let H be the set $\{v \in E^0 \mid \mathbf{1}_v \in I\}$. It follows from the ψ_E-invariance of I that H is hereditary (that is, whenever $e \in E^1$ with $s(e) \in H$, then $r(e) \in H$). Let $v \in H$. Then $\eta(\mathbf{1}_v) = \sum_{\alpha\in K} f_\alpha \mathbf{1}_\alpha$ for some non-empty finite subset $K \subseteq E^n$ and non-zero elements $f_\alpha \in F$, $\alpha \in K$. Since $\mathbf{1}_v\eta(\mathbf{1}_v)\mathbf{1}_v = \eta(\mathbf{1}_v\mathbf{1}_v\mathbf{1}_v) = \eta(\mathbf{1}_v)$, it follows that $r(\alpha) = s(\alpha) = v$ for each $\alpha \in K$. Let $\alpha \in E^n$ with $r(\alpha) = s(\alpha) = v$. Since

$$(\psi_E)_n\big(\mathbf{1}_{\bar\alpha} \otimes \eta(\mathbf{1}_v)\big)\mathbf{1}_\alpha = \eta\big((\psi_E)_n(\mathbf{1}_{\bar\alpha}\mathbf{1}_v \otimes \mathbf{1}_\alpha)\big) = \eta(\mathbf{1}_v),$$

it follows that $K \subseteq \{\alpha\}$. Hence it must be the case that there is exactly one $\alpha_v \in E^n$ with $r(\alpha) = s(\alpha) = v$, and that K consists of this element. Thus there is for each $v \in H$ a unique $\alpha_v \in E^n$ with $r(\alpha) = s(\alpha) = v$ and $\eta(\mathbf{1}_v) = f_{\alpha_v}\mathbf{1}_{\alpha_v}$ for some $f_{\alpha_v} \in F \setminus \{0\}$.

Let $v \in H$, let $\alpha_v = (e_1, e_2, \dots, e_n)$ and assume that there is an $e \in E^1 \setminus \{e_1\}$ with $s(e) = v$. Then

$$\eta(\mathbf{1}_{r(e)}) = \eta\big(\psi_E(\mathbf{1}_{\bar e}\mathbf{1}_v \otimes \mathbf{1}_e)\big) = S_{\mathbf{1}_{\bar e}} T_{\eta(\mathbf{1}_v)}\mathbf{1}_e = f_{\alpha_v} S_{\mathbf{1}_{\bar e}} T_{\mathbf{1}_{\alpha_v}}\mathbf{1}_e = 0$$

which contradicts the fact that η is injective. Thus, for each $v \in H$ it is the case that $vE^1 = \{e_1\}$ where e_1 is the initial part of α_v. It follows that every $v \in H$ is the base of a closed path which has no exit. In particular, (E^0, E^1, r, s) has a closed path which has no exit.

On the other hand, it is straightforward to check that if $\alpha_v = (e_1, e_2, \dots, e_n)$ is a closed path without an exit, then $H = \{s(e_i) \mid i \in \{1, 2, \dots, n\}\}$ is a hereditary subset of E^0, $I = \operatorname{span}_F\{\mathbf{1}_v \mid v \in H\}$ is contained in $J_E^{[\infty]}$ (cf. Example 3.5) and is a ψ_E-invariant ideal in R_E, and the F-linear map $\eta : I \to Q_E^{\otimes n}$ given by $\mathbf{1}_{s(e_i)} \mapsto \mathbf{1}_{(e_i,e_{i+1},\dots,e_n,e_1,e_2,\dots,e_{i-1})}$ for $i \in \{1, 2, \dots, n\}$ is an injective R_E-bimodule homomorphism $\eta : I \to Q_E^{\otimes n}$ satisfying $S_p T_{\eta(x)}(q) = \eta(\psi_E(px \otimes q))$ for $p \in P_E$, $x \in I$ and $q \in Q_E$.

Thus J_E satisfies condition (L) with respect to the R_E-system (P_E, Q_E, ψ_E) if and only if every closed path in (E^0, E^1, r, s) has an exit (cf. [19, Definition 6.3]).

5. The Cuntz–Krieger uniqueness theorem

In this section the *Cuntz–Krieger uniqueness property* is defined (Definition 5.1), and the *Cuntz–Krieger uniqueness result* is proved (Theorem 5.2).

Recall from [5, Definition 1.2] that if (S', T', σ', B) is a covariant representation of (P, Q, ψ), then $\mathcal{R}\langle S', T', \sigma'\rangle$ is defined to be the subring of B generated by $\sigma'(R) \cup T'(Q) \cup S'(P)$.

Definition 5.1. We say that the ideal J has the *Cuntz–Krieger uniqueness property* with respect to the R-system (P, Q, ψ) if the following holds:

> If $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ are two injective covariant representations of (P, Q, ψ) and they are both Cuntz–Pimsner invariant relative to J, then there is a ring isomorphism ϕ between $\mathcal{R}\langle S_1, T_1, \sigma_1\rangle$ and $\mathcal{R}\langle S_2, T_2, \sigma_2\rangle$ such that $\phi \circ \sigma_1 = \sigma_2$, $\phi \circ S_1 = S_2$ and $\phi \circ T_1 = T_2$.

We will often, when it is clear from the context which R-system (P, Q, ψ) we are working with, simply say that J has the Cuntz–Krieger uniqueness property instead of saying that it has the Cuntz–Krieger uniqueness property with respect to (P, Q, ψ).

Recall from [5, Definition 4.6] that J is said to be a *maximal faithful, ψ-compatible* ideal if $J = J'$ for any faithful, ψ-compatible ideal J' in R satisfying $J \subseteq J'$.

Theorem 5.2. *The following* 5 *conditions are equivalent*:

(1) *The ideal J has the Cuntz–Krieger uniqueness property.*
(2) *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J, then the ring homomorphism*

$$\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$$

from [5, Theorem 3.18] is injective.
(3) *The subring $\sigma(R)$ has the ideal intersection property.*
(4) *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property, and J is a maximal faithful, ψ-compatible ideal.*
(5) *The ideal J satisfies condition (L) and is a maximal faithful, ψ-compatible ideal.*

Proof. (1) $\Rightarrow$ (2): The ring homomorphism $\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ is the unique ring homomorphism from $\mathcal{O}_{(P,Q,\psi)}(J)$ to B such that $\eta^J_{(S',T',\sigma',B)} \circ \sigma = \sigma'$, $\eta^J_{(S',T',\sigma',B)} \circ S = S'$ and $\eta^J_{(S',T',\sigma',B)} \circ T = T'$, so it follows by assumption that $\eta^J_{(S',T',\sigma',B)}$ is injective.

(2) $\Rightarrow$ (1): If $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ are two injective covariant representations of (P, Q, ψ) and there are both Cuntz–Pimsner invariant relative to J, then $\phi = \eta^J_{(S_2,T_2,\sigma_2,B_2)} \circ (\eta^J_{(S_1,T_1,\sigma_1,B_1)})^{-1}$ is a ring isomorphism between $\mathcal{R}\langle S_1, T_1, \sigma_1\rangle$ and $\mathcal{R}\langle S_2, T_2, \sigma_2\rangle$ such that $\phi \circ \sigma_1 = \sigma_2$, $\phi \circ S_1 = S_2$ and $\phi \circ T_1 = T_2$.

(2) $\Rightarrow$ (3): Let K be an ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ such that $K \cap \sigma(R) = \{0\}$, and let $\wp : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{(P,Q,\psi)}(J)/K$ be the quotient map. Then $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/K)$ is an injective and surjective covariant representation of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J. It follows by assumption that $\wp = \eta_{(\wp\circ S,\wp\circ T\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/K)}$ is injective. Thus $K = \{0\}$, which proves that $\sigma(R)$ has the ideal intersection property.

(3) $\Rightarrow$ (4): Since $\sigma(R) \subseteq \mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$, it follows that $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property if $\sigma(R)$ has. If J is not a maximal faithful, ψ-invariant ideal, then there exists a faithful, ψ-compatible ideal J' such that $J \subsetneq J'$. It follows from [5, Remark 4.1] that $\rho_J(\mathcal{T}(J'))$ then would be a non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ with a zero intersection with $\sigma(R)$, which would mean that $\sigma(R)$ does not have the ideal intersection property. Thus it must be the case that J is a maximal faithful, ψ-invariant ideal.

(4) $\Rightarrow$ (2): Since J is a maximal faithful, ψ-compatible ideal by assumption, it follows that $J_{(S',T',\sigma',B)} = J$. Thus it follows from Theorem 4.2 that $\eta^J_{(S',T',\sigma',B)}$ is injective.

(4) $\Leftrightarrow$ (5) follows from Theorem 4.2. $\square$

Example 5.3. Let E be a directed graph, F a field, and let (P_E, Q_E, ψ_E) and J_E be as in Example 2.1. It follows from [5, Lemma 5.2 and Example 5.8] that J_E is a maximal faithful, ψ_E-compatible ideal. We therefore recover [19, Theorem 6.8 and Corollary 6.10] from [5, Example 5.8], Example 4.3 and Theorem 5.2.

6. Condition (K)

In this section condition (K) is introduced (Definition 6.1), and sufficient and necessary conditions for when every ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ is graded are given (Theorem 6.2).

Recall from [5, Section 7] that if I is a ψ-invariant ideal in R, then $R_I = R/I$, $Q_I = Q/QI$ and ${}_IP = P/IP$, and $\wp_I$ denotes the corresponding quotient maps. Recall also that there is an R_I-bimodule homomorphism $\psi_I : {}_IP \otimes Q_I \to R_I$ given by $\psi_I(\wp_I(p) \otimes \wp_I(q)) = \wp_I(\psi(p \otimes q))$. The triple $({}_IP, Q_I, \psi_I)$ is then an R_I-system satisfying condition (FS) (see [5, Lemma 7.4]). Recall from [5, Definition 7.5] that a T*-pair* is a pair (I, J') where I and J' are ideals in R such that $I \subseteq J$, I is ψ-invariant, and $J'_I := \wp_I(J')$ is a faithful, ψ_I-compatible ideal in R_I.

Definition 6.1. We say that the ideal J satisfies *condition* (K) with respect to the R-system (P, Q, ψ) if J'_I satisfies condition (L) with respect to the R_I-system $({}_IP, Q_I, \psi_I)$ whenever (I, J') is a T-pair of (P, Q, ψ) such that $J \subseteq J'$.

We will often, when it is clear from the context which R-system (P, Q, ψ) we are working with, simply say that J satisfies condition (K) instead of saying that it satisfies condition (K) with respect to (P, Q, ψ).

Theorem 6.2. *The following* 3 *conditions are equivalent*:

(1) *Every ideal of* $\mathcal{O}_{(P,Q,\psi)}(J)$ *is graded.*
(2) *The ideal* J *satisfies condition* (K).
(3) *If* (S', T', σ', B) *is a covariant representation of* (P, Q, ψ) *which is Cuntz–Pimsner invariant relative to* J, *and* $(I, J') = \omega_{(S',T',\sigma',B)}$ (*see* [5, *Proposition* 7.8]), *then the ring homomorphism*

$$\eta^{(I,J')}_{(S',T',\sigma',B)} : \mathcal{O}_{({}_IP,Q_I,\psi_I)}\big(J'_I\big) \to B$$

from [5, *Theorem* 7.11(ii)] *is injective.*

Proof. (1) $\Rightarrow$ (2): Let $\omega = (I, J')$ be a T-pair of (P, Q, ψ) such that $J \subseteq J'$ and let H be a non-zero ideal in $\mathcal{O}_{({}_IP,Q_I,\psi_I)}(J'_I)$. Recall from [5, p. 36] that there is a covariant representation $(\iota^\omega_P, \iota^\omega_Q, \iota^\omega_R, \mathcal{O}_{({}_IP,Q_I,\psi_I)}(J'_I))$ such that $\iota^\omega_P = \iota^{J'_I}_{{}_IP} \circ \wp_I$, $\iota^\omega_Q = \iota^{J'_I}_{Q_I} \circ \wp_I$ and $\iota^\omega_R = \iota^{J'_I}_{R_I} \circ \wp_I$. It follows from [5, Remark 3.25 and Theorem 3.29] that there is a surjective, graded ring homomorphism $\phi : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{({}_IP,Q_I,\psi_I)}(J'_I)$ which intertwines the two representations $(S, T, \sigma, \mathcal{O}_{(P,Q,\psi)}(J))$ and $(\iota^\omega_P, \iota^\omega_Q, \iota^\omega_R, \mathcal{O}_{({}_IP,Q_I,\psi_I)}(J'_I))$. We then have that $\phi^{-1}(H)$ is an ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$. Thus $\phi^{-1}(H)$ is graded by assumption. It follows that also H is graded. It therefore follows from Theorem 4.2 that J'_I satisfies condition (L) with respect to the R_I-system $({}_IP, Q_I, \psi_I)$. This proves that J satisfies condition (K).

(2) $\Rightarrow$ (3): It follows from [5, Lemma 7.10] that there is an injective covariant representation (S_I, T_I, σ_I, B) of $({}_IP, Q_I, \psi_I)$ such that $S_I = S' \circ \wp_I$, $T_I = T' \circ \wp_I$ and $\sigma_I = \sigma' \circ \wp_I$. Since $\pi_{T_I,S_I}(\mathcal{F}_{{}_IP}(Q_I)) = \pi_{T',S'}(\mathcal{F}_P(Q))$, it follows that $J_{(S_I,T_I,\sigma_I,B)} = \wp_I(J_{(S',T',\sigma',B)}) = \wp_I(J') = J'_I$. It therefore follows from Theorem 4.2 that $\eta^{(I,J')}_{(S',T',\sigma',B)} = \eta^{J'_I}_{(S_I,T_I,\sigma_I,B)}$ is injective.

(3) $\Rightarrow$ (1): Let H be an ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ and $\wp : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{(P,Q,\psi)}(J)/H$ be the quotient map. Then $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/H)$ is a covariant representation which is Cuntz–Pimsner invariant relative to J. Let $(I, J') = \omega_{(\wp\circ S,\wp\circ T,\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/H)}$. Then $\eta^{(I,J')}_{(\wp\circ S,\wp\circ T,\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/H)}$ is injective by assumption. Since $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}^{(n)}_{({}_IP,Q_I,\psi_I)}(J'_I)$ is a $\mathbb{Z}$-grading of $\mathcal{O}_{({}_IP,Q_I,\psi_I)}(J'_I)$, it follows that

$$\bigoplus_{n\in\mathbb{Z}} \wp\big(\mathcal{O}^{(n)}_{(P,Q,\psi)}(J)\big) = \bigoplus_{n\in\mathbb{Z}} \eta^{(I,J')}_{(\wp\circ S,\wp\circ T,\wp\circ\sigma,\mathcal{O}_{(P,Q,\psi)}(J)/H)}\big(\mathcal{O}^{(n)}_{({}_IP,Q_I,\psi_I)}(J'_I)\big)$$

is a $\mathbb{Z}$-grading of $\mathcal{O}_{(P,Q,\psi)}(J)/H$. Thus H is graded. $\square$

Remark 6.3. It follows from the above theorem that if J satisfies condition (K), then [5, Theorem 7.27] gives a bijective correspondence between the set of all ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$ and the set of T-pairs (I, J') of (P, Q, ψ) satisfying $J \subseteq J'$.

Example 6.4. Let E be a directed graph, F a field, and let (P_E, Q_E, ψ_E) and J_E be as in Example 2.1. By combining [19, Theorem 5.7 and Proposition 6.12] and [5, Example 7.31] with the characterization given in Example 4.3 of when J_E satisfies condition (L), one sees that J_E satisfies condition (K) with respect to the R_E-system (P_E, Q_E, ψ_E) if and only if every $v \in E^0$ is either the base of no closed path or the base of at least two simple closed paths (cf. [19, Definition 6.11]). We therefore recover [19, Theorem 6.16 and Corollary 6.17] from [5, Examples 5.8 and 7.31], Theorem 6.2 and Remark 6.3.

7. Simplicity of $\mathcal{O}_{(P,Q,\psi)}(J)$

In this section sufficient and necessary conditions for when $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple are given (Theorem 7.3).

Definition 7.1. We say that J is a *super maximal* ψ-compatible ideal if the only T-pairs (I, J') of (P, Q, ψ) which satisfies that $J \subseteq J'$, are $(0, J)$ and (R, R).

Since $(0, J')$ is a T-pair of (P, Q, ψ) for any faithful ψ-compatible ideal J' in R, it follows that if J is a super maximal ψ-compatible ideal, then it is also a maximal ψ-compatible ideal.

Remark 7.2. It follows from [5, Theorem 7.27] that J is a super maximal ψ-compatible ideal if and only if the only graded ideals in $\mathcal{O}_{(P,Q,\psi)}(J)$ are $\{0\}$ and $\mathcal{O}_{(P,Q,\psi)}(J)$.

Theorem 7.3. *The following* 5 *conditions are equivalent*:

(1) *The ring $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple.*
(2) *The subring $\sigma(R)$ has the ideal intersection property and J is a super maximal ψ-compatible ideal.*
(3) *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property and J is a super maximal ψ-compatible ideal.*
(4) *The ideal J satisfies condition (L) and is a super maximal ψ-compatible ideal.*
(5) *If (S', T', σ', B) is a non-zero covariant representation of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J, then the ring homomorphism*

$$\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$$

from [5, *Theorem* 3.18] *is injective.*

Proof. (1) $\Rightarrow$ (2): If $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple, then clearly $\sigma(R)$ has the ideal intersection property. If (I, J') is a T-pair of (P, Q, ψ) different from $(0, J)$, then it follows from [5, Theorem 7.27] that $H^J_{(I,J')}$ is a non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$. If $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple, then that would imply that $H^J_{(I,J')} = \mathcal{O}_{(P,Q,\psi)}(J)$ and thus $(I, J') = (R, R)$ from which it follows that J is a super maximal ψ-compatible ideal.

(2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) follow from Theorem 5.2 and the fact that J is a maximal ψ-compatible ideal if it is a super maximal ψ-compatible ideal.

(2) $\Rightarrow$ (5): It follows from [5, Proposition 7.8] that $(I_{(S',T',\sigma',B)}, J_{(S',T',\sigma',B)})$ is a T-pair. Since (S', T', σ', B) is Cuntz–Pimsner invariant relative to J, it follows from [5, Remark 3.25] that $J \subseteq J_{(S',T',\sigma',B)}$, and since (S', T', σ', B) is non-zero, it follows from [5, Theorem 7.11] that $(I_{(S',T',\sigma',B)}, J_{(S',T',\sigma',B)}) \neq (R, R)$. Thus $(I_{(S',T',\sigma',B)}, J_{(S',T',\sigma',B)}) = (0, J)$ which implies that (S', T', σ', B) is an injective representation. It then follows from Theorem 5.2 that $\eta^J_{(S',T',\sigma',B)}$ is injective.

(5) $\Rightarrow$ (1): Let K be a proper ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$, and let $\wp : \mathcal{O}_{(P,Q,\psi)}(J) \to \mathcal{O}_{(P,Q,\psi)}(J)/K$ be the quotient map. Then $(\wp \circ S, \wp \circ T, \wp \circ \sigma, \mathcal{O}_{(P,Q,\psi)}(J)/K)$ is a surjective covariant representation of (P, Q, ψ) which is Cuntz–Pimsner invariant relative to J. It follows by assumption that $\wp = \eta_{(\wp\circ S, \wp\circ T \wp\circ\sigma, \mathcal{O}_{(P,Q,\psi)}(J)/K)}$ is injective. Thus $K = \{0\}$ which proves that $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple. $\square$

Example 7.4. Let E be a directed graph, F a field, and let (P_E, Q_E, ψ_E) and J_E be as in Example 2.1. It follows from [5, Example 5.8], [19, Theorem 5.7] (cf. [5, Example 7.31]) and Remark 7.2 that J_E is super maximal ψ_E-compatible ideal if and only if the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0. Thus we recover [19, Theorem 6.18] from [5, Example 5.8], Theorem 7.3 and the characterization given in Example 4.3 of when J_E satisfies condition (L).

8. Toeplitz rings

When $J = \{0\}$, then $\mathcal{O}_{(P,Q,\psi)}(J)$ is the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)}$ introduced in [5, Theorem 1.7], and J automatically satisfies condition (L). Thus the following 3 corollaries follow from Theorems 4.2, 5.2 and 7.3, respectively.

Corollary 8.1. *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ), then the ring homomorphism $\eta_{(S',T',\sigma',B)} : \mathcal{T}_{(P,Q,\psi)} \to B$ from [5, Theorem 1.7] is injective if and only if $J_{(S',T',\sigma',B)} = \{0\}$.*

Corollary 8.2. *Assume that there are no non-zero faithful ψ-compatible ideals of R. If $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ are two injective covariant representations of (P, Q, ψ), then there is a ring isomorphism ϕ between $\mathcal{R}\langle S_1, T_1, \sigma_1\rangle$ and $\mathcal{R}\langle S_2, T_2, \sigma_2\rangle$ such that $\phi \circ \sigma_1 = \sigma_2$, $\phi \circ S_1 = S_2$ and $\phi \circ T_1 = T_2$.*

Corollary 8.3. *The Toeplitz ring $\mathcal{T}_{(P,Q,\psi)}$ is simple if and only if $(0, 0)$ and (R, R) are the only T-pairs of (P, Q, ψ).*

As an application of Corollary 8.2 we will now present an algebraic analogue of the uniqueness theorem for the Toeplitz C^*-algebra of a directed graph given in [6, Theorem 4.1]. We begin with a definition.

Definition 8.4. Let $E = (E^0, E^1, r, s)$ be a directed graph, let F be a field and B an F-algebra. A *Toeplitz–Cuntz–Krieger E-family* in B consists of a family $\{p_v \mid v \in E^0\}$ of pairwise orthogonal idempotents in B together with a family $\{x_e, y_e \mid e \in E^1\}$ of elements in B satisfying the following relations

(a) $p_{s(e)} x_e = x_e = x_e p_{r(e)}$ for $e \in E^1$,
(b) $p_{r(e)} y_e = y_e = y_e p_{s(e)}$ for $e \in E^1$,
(c) $y_e x_f = \delta_{e,f} p_{r(e)}$ for $e, f \in E^1$,

where $\delta_{e,f}$ denotes the Kronecker delta function.

We are now ready to state and prove our algebraic analogue of [6, Theorem 4.1].

Theorem 8.5. *Let $E=(E^0,E^1,r,s)$ be a directed graph and let F be a field. Let R_E be the ring and (P_E,Q_E,ψ_E) the R_E-system associated to E in Example 2.1 and let $(\iota_{P_E},\iota_{Q_E},\iota_{R_E},\mathcal{T}_{(P_E,Q_E,\psi_E)})$ be the Toeplitz representation of (P_E,Q_E,ψ_E). Then the following statements are true.*

1. *The family $\{\iota_{R_E}(\mathbf{1}_v)\mid v\in E^0\}$ together with $\{\iota_{Q_E}(\mathbf{1}_e),\iota_{P_E}(\mathbf{1}_{\bar{e}})\mid e\in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family.*
2. *If B is an F-algebra and $\{p_v\mid v\in E^0\}$ together with $\{x_e,y_e\mid e\in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family, then there exists a unique F-algebra homomorphism $\eta:\mathcal{T}_{(P_E,Q_E,\psi_E)}\to B$ satisfying $\eta(\iota_{R_E}(\mathbf{1}_v))=p_v$ for $v\in E^0$, and $\eta(T(\mathbf{1}_e))=x_e$ and $\eta(S(\mathbf{1}_{\bar{e}}))=y_e$ for $e\in E^1$.*
3. *The homomorphism η is injective if and only if $p_v\neq 0$ for each $v\in E^0$ and $p_v\neq\sum_{e\in vE^1}x_ey_e$ for $v\in E^0$ with $0<|vE^1|<\infty$.*

Proof. That $\mathcal{T}_{(P_E,Q_E,\psi)}$ is an F-algebra and that $\{\iota_{R_E}(\mathbf{1}_v)\mid v\in E^0\}\cup\{\iota_{Q_E}(\mathbf{1}_e),\iota_{P_E}(\mathbf{1}_{\bar{e}})\mid e\in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family is proved in [5, Example 1.10]. It is also proved in [5, Example 1.10] that if B is an F-algebra and $\{p_v\mid v\in E^0\}$ together with $\{x_e,y_e\mid e\in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family, then there is a covariant representation (S',T',σ',B) of (P_E,Q_E,ψ) such that $S'(\lambda\mathbf{1}_{\bar{e}})=\lambda y_e$ and $T'(\lambda\mathbf{1}_e)=\lambda x_e$ for $e\in E^1$ and $\lambda\in F$, and $\sigma'(\lambda\mathbf{1}_v)=\lambda p_v$ for $v\in E^0$ and $\lambda\in F$. It then follows from [5, Theorem 1.7] that there is a ring homomorphism $\eta:\mathcal{T}_{(P_E,Q_E,\psi)}\to B$ such that $\eta(\iota_{R_E}(\lambda\mathbf{1}_v))=\sigma'(\lambda\mathbf{1}_v)=\lambda p_v$ for $v\in E^0$ and $\lambda\in F$, and $\eta(\iota_{Q_E}(\lambda\mathbf{1}_e))=T'(\lambda\mathbf{1}_e)=\lambda x_e$ and $\eta(\iota_{P_E}(\lambda\mathbf{1}_{\bar{e}}))=S'(\lambda\mathbf{1}_{\bar{e}})=\lambda y_e$ for $e\in E^1$ and $\lambda\in F$. It follows that η is an F-algebra homomorphism and that $\eta(\iota_{R_E}(\mathbf{1}_v))=p_v$ for $v\in E^0$, and $\eta(\iota_{Q_E}(\mathbf{1}_e))=x_e$ and $\eta(\iota_{P_E}(\mathbf{1}_{\bar{e}}))=y_e$ for $e\in E^1$. Since $\mathcal{T}_{(P_E,Q_E,\psi)}$ is generated, as an F-algebra, by $\{\iota_{R_E}(\mathbf{1}_v)\mid v\in E^0\}\cup\{\iota_{Q_E}(\mathbf{1}_e),\iota_{P_E}(\mathbf{1}_{\bar{e}})\mid e\in E^1\}$, there cannot be any other F-algebra homomorphism from $\mathcal{T}_{(P_E,Q_E,\psi)}$ to B which for every $v\in E^0$ maps $\iota_{R_E}(\mathbf{1}_v)$ to p_v and for any $e\in E^1$ maps $\iota_{Q_E}(\mathbf{1}_e)$ to x_e and $\iota_{P_E}(\mathbf{1}_{\bar{e}})$ to y_e.

The map ι_{R_E} is injective by [5, Theorem 1.7]. It follows that if η is injective, then $p_v\neq 0$ for each $v\in E^0$. Assume that $p_v\neq 0$ for each $v\in E^0$. Since $R_E=\bigoplus_{v\in E^0}R_v$ where each R_v is a copy of F, it follows that σ' is injective. Thus it follows from Corollary 8.2 that η is injective if and only if $J_{(S',T',\sigma',B)}=0$. It follows from [5, Lemma 3.24] that

$$J_{(S',T',\sigma',B)}=\bigl\{r\in\Delta^{-1}\bigl(\mathcal{F}_{P_E}(Q_E)\bigr)\bigm|\sigma'(r)=\pi_{T',S'}\bigl(\Delta(r)\bigr)\bigr\}.$$

It is proved in [5, Example 5.8] that

$$\Delta^{-1}\bigl(\mathcal{F}_{P_E}(Q_E)\bigr)=\operatorname{span}_F\bigl\{\mathbf{1}_v\bigm|0<\bigl|vE^1\bigr|<\infty\bigr\},$$

and is straightforward to check that $\Delta(\mathbf{1}_v)=\sum_{e\in vE^1}\theta_{\mathbf{1}_e,\mathbf{1}\bar{e}}$ if $\mathbf{1}_v\in\Delta^{-1}(\mathcal{F}_{P_E}(Q_E))$. It follows that

$$J_{(S',T',\sigma',B)}=\operatorname{span}_F\Biggl\{\mathbf{1}_v\Biggm|0<\bigl|vE^1\bigr|<\infty,\ p_v=\sum_{e\in vE^1}x_ey_e\Biggr\}.$$

Thus η is injective if and only if $p_v\neq 0$ for each $v\in E^0$ and $p_v\neq\sum_{e\in vE^1}x_ey_e$ for $v\in E^0$ with $0<|vE^1|<\infty$. $\square$

9. Crossed products of a ring by an automorphism and fractional skew monoid rings of a corner isomorphism

We will in this section use Theorem 7.3 to give a characterization of when the fractional skew monoid ring of a corner isomorphism is simple (Corollary 9.8), and when the crossed product of a ring by an automorphism is simple (Corollary 9.9). We end the section with two simple examples which illustrate how these characterizations can be used.

A ring R has *local units* if given any finite set $F \subseteq R$ there exists an idempotent $e \in R$ such that $er = re = r$ for every $r \in F$, in other words, the set of all idempotents of R, $\mathrm{Idem}(R)$, is a directed system (with order $e \leqslant f$ if and only if $ef = fe = e$) and $R = \bigcup_{e \in \mathrm{Idem}(R)} eRe$.

Let R be a ring with local units and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R)R\alpha(R) \subseteq \alpha(R)$ (notice this is equivalent to $\alpha(R)R\alpha(R) = \alpha(R)$ since R has local units). Recall from [5, Example 5.6] that if P is the R-bimodule which is equal to $\operatorname{span}\{r_1\alpha(r_2) \mid r_1, r_2, \in R\}$ as a set, has the additive structure it inherits from R, and has the left and right actions given by $r \cdot p = rp$ and $p \cdot r = p\alpha(r)$ for $r \in R$ and $p \in P$; Q is the R-bimodule which is equal to $\operatorname{span}\{\alpha(r_1)r_2 \mid r_1, r_2 \in R\}$ as a set, has the additive structure it inherits from R, and has the left and right given by $r \cdot q = \alpha(r)q$ and $q \cdot r = qr$ for $r \in R$ and $q \in Q$; and $\psi : P \otimes Q \to R$ is the R-bimodule homomorphism given by $p \otimes q \mapsto pq$, then (P, Q, ψ) is an R-system. Recall also that R is a uniquely maximal, faithful, ψ-compatible ideal and that if α is an automorphism, then $\mathcal{O}_{(P,Q,\psi)}(R)$ is isomorphic to the crossed product $R \times_\alpha \mathbb{Z}$ of R by α. If R is unital, and we let $e = \alpha(1)$ (where 1 denotes the unit of R), then e is an idempotent and $\alpha(R) = \alpha(R)R\alpha(R) = eRe$. It follows from [5, Example 5.7] that we in this case have that $\mathcal{O}_{(P,Q,\psi)}(R)$ is isomorphic to the fractional skew monoid ring $R[t_+, t_-; \alpha]$ that Ara, González-Barroso, Goodearl and Pardo have constructed in [3]. We will use these facts together with Theorem 7.3 to give a characterization of when the crossed product $R \times_\alpha \mathbb{Z}$ is simple and when the fractional skew monoid ring $R[t_+, t_-; \alpha]$ is simple, but first we introduce some notions and results that we will use for this.

Unless otherwise stated, α will just be assumed to be an injective ring homomorphism such that $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. We let (P, Q, ψ) be the R-system defined above. Using that R has local units, it is not difficult to see that for $n \in \mathbb{N}$, the R-bimodule $P^{\otimes n}$ is isomorphic to the R-bimodule which is equal to $\operatorname{span}\{r_1\alpha^n(r_2) \mid r_1, r_2, \in R\}$ as a set, has the additive structure it inherits from R, and has the left and right actions given by $r \cdot p = rp$ and $p \cdot r = p\alpha^n(r)$, respectively. Likewise, $Q^{\otimes n}$ is isomorphic to the R-bimodule which is equal to $\operatorname{span}\{\alpha^n(r_1)r_2 \mid r_1, r_2 \in R\}$ as a set, has the additive structure it inherits from R and has the left and right given by $r \cdot q = \alpha^n(r)q$ and $q \cdot r = qr$, respectively. We will simply identify $P^{\otimes n}$ and $Q^{\otimes n}$ with these two R-bimodules. We will use a $\cdot$ to indicate the left and right actions of R on $P^{\otimes n}$ and $Q^{\otimes n}$ to distinguish these actions from the ordinary multiplication in R. It is straightforward to check that if $q \in Q$, $q_n \in Q^{\otimes n}$ and $p \in P$, then $S_p T_{q_n}(q) = \alpha^n(p)\alpha(q_n)q$. Let $(S, T, \sigma, \mathcal{O}_{(P,Q,\psi)}(R))$ denote the Cuntz–Pimsner representation of (P, Q, ψ) relative to R. Then $S^n(p_n)\sigma(r) = S^n(p_n\alpha^n(r))$, $\sigma(r)S^n(p_n) = S^n(rp_n)$, $S^n(p_n)S^{n'}(p'_{n'}) = S^{n+n'}(p_n\alpha^n(p'_{n'}))$, $T^n(q_n)\sigma(r) = T^n(q_n r)$, $\sigma(r)T^n(q_n) = T^n(\alpha^n(r)q_n)$, $T^n(q_n)T^{n'}(q'_{n'}) = T^{n+n'}(\alpha^{n'}(q_n)q'_{n'})$, $S^n(p_n)T^n(q_n) = \sigma(p_n q_n)$ and $T^n(q_n)S^n(p_n) = \sigma(\alpha^{-n}(q_n p_n))$ for $n, n' \in \mathbb{N}$, $p_n \in P^n$, $r \in R$, $p_{n'} \in P^{\otimes n'}$, $q_n \in Q^{\otimes n}$ and $q_{n'} \in Q^{\otimes n'}$ where p_n, $p_{n'}$, q_n and $q_{n'}$ are considered as elements of R and the multiplication of R is used. It follows that $\mathcal{O}_{(P,Q,\psi)}(R)^{(0)} = \sigma(R)$, and that $\mathcal{O}_{(P,Q,\psi)}(R)^{(n)} = T^n(Q^{\otimes n})$ and $\mathcal{O}_{(P,Q,\psi)}(R)^{(-n)} = S^n(P^{\otimes n})$ for $n \in \mathbb{N}$.

We say that an ideal I of R is *strongly α-invariant* if $\alpha^{-1}(I) = I$.

Proposition 9.1. *Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$, and let (P, Q, ψ) be the R-system defined above. Then there is a bijective correspondence between graded ideals of $\mathcal{O}_{(P,Q,\psi)}(R)$ and strongly α-invariant ideals of R.*

Proof. For each strongly α-invariant ideal I in R, let H_I be the ideal in $\mathcal{O}_{(P,Q,\psi)}(R)$ generated by $\sigma(I)$; and let for each graded ideal H in $\mathcal{O}_{(P,Q,\psi)}(R)$, $I_H = \{x \in R \mid \sigma(x) \in H\}$. We will show that H_I is a graded ideal in $\mathcal{O}_{(P,Q,\psi)}(R)$, that I_H is a strongly α-invariant ideal in R, and that $I_{H_I} = I$ and $H_{I_H} = H$ for all strongly α-invariant ideals I in R and all graded ideals H in $\mathcal{O}_{(P,Q,\psi)}(R)$. This will establish the bijective correspondence between the graded ideals of $\mathcal{O}_{(P,Q,\psi)}(R)$ and the strongly α-invariant ideals of R.

Let I be a strongly α-invariant ideal in R. It is not difficult to check that if we let $H^{(0)} = \sigma(I)$ and for each $n \in \mathbb{N}$ let $H^{(n)} = \operatorname{span}\{T^n(\alpha^n(r)x) \mid r \in R,\ x \in I\}$ and $H^{(-n)} = \operatorname{span}\{S^n(x\alpha^n(r)) \mid x \in I,\ r \in R\}$, then $\bigoplus_{n \in \mathbb{Z}} H^{(n)}$ is an ideal in $\mathcal{O}_{(P,Q,\psi)}(R)$. Since $\bigoplus_{n \in \mathbb{Z}} H^{(n)}$ contains $\sigma(I)$ and itself must be contained in any ideal which contains $\sigma(I)$, it must be the case that $H_I = \bigoplus_{n\in\mathbb{Z}} H^{(n)}$. It follows that H_I is graded and that $I_{H_I} = I$.

Let H be a graded ideal in $\mathcal{O}_{(P,Q,\psi)}(R)$. It is clear that I_H is an ideal in R. Let $x \in I_H$. Choose idempotents $e_1, e_2 \in R$ such that $e_1\alpha(x)e_1 = \alpha(x)$ and $e_2xe_2 = x$. Then

$$\sigma\big(\alpha(x)\big) = S\big(e_1\alpha(e_2)\big)\sigma(x)T\big(\alpha(e_2)e_1\big) \in H,$$

so $\alpha(x) \in I_H$. Let $y \in R$ such that $\alpha(y) \in I_H$. Choose an idempotent $f_1 \in R$ such that $f_1yf_1 = y$ and an idempotent $f_2 \in R$ such that $\alpha(f_1)f_2 = f_2\alpha(f_1) = \alpha(f_1)$. Then

$$\sigma(y) = \sigma(f_1yf_1) = T\big(\alpha(f_1)f_2\big)\sigma\big(\alpha(y)\big)S\big(f_2\alpha(f_1)\big) \in H,$$

so $y \in I_H$. This shows that I_H is a strongly α-invariant ideal in R. Since $\mathcal{O}_{(P,Q,\psi)}(R)^{(0)} = \sigma(R)$, it follows from [5, Lemma 3.35] that H is generated by $\sigma(I_H)$. Thus $H = H_{I_H}$. $\square$

By combining the above result with Remark 7.2 we get the following characterization of when R is a super maximal ψ-compatible ideal.

Corollary 9.2. *Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$, and let (P, Q, ψ) be the R-system defined above. Then the following three conditions are equivalent*:

(1) *The ring R is a super maximal ψ-compatible ideal.*
(2) *The only graded ideals in $\mathcal{O}_{(P,Q,\psi)}(R)$ are $\{0\}$ and $\mathcal{O}_{(P,Q,\psi)}(R)$.*
(3) *The only strongly α-invariant ideals in R are $\{0\}$ and R.*

We next introduce the *multiplier ring* of R (see for example [4]). A double centralizer on R is a pair (f, g) where $f : R \to R$ is a right R-module homomorphism and $g : R \to R$ is a left R-module homomorphism satisfying $r_1 f(r_2) = g(r_1)r_2$ for all $r_1, r_2 \in R$. The *multiplier ring* of R is the ring $\mathcal{M}(R)$ of all double centralizers on R with addition defined by $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and product defined by $(f_1, g_1)(f_2, g_2) = (f_1 \circ f_2, g_2 \circ g_1)$. Notice that $(\mathrm{Id}_R, \mathrm{Id}_R)$ is a unit of $\mathcal{M}(R)$. There is a ring homomorphism $\iota : R \to \mathcal{M}(R)$ given by $\iota(r) = (f_r, g_r)$ where $f_r(s) = rs$ and $g_r(s) = sr$ for $r, s \in R$. Since R has local units, ι is injective. We will therefore simple regard R as a subring of $\mathcal{M}(R)$. We then have that if $u = (f, g) \in \mathcal{M}(R)$ and $r \in R$, then $ur = f(r)$ and $ru = g(r)$. It follows that R is an ideal in $\mathcal{M}(R)$. Notice that $R = \mathcal{M}(R)$ if and only if R is unital.

Definition 9.3. Let $n \in \mathbb{N}$ and let R be a ring with local units. A ring homomorphism $\alpha : R \to R$ is said to be *inner with periodicity* n if there exist $u, v \in \mathcal{M}(R)$ such that $vu = 1$ (where 1 denotes the unit of $\mathcal{M}(R)$), and $\alpha^n(r) = urv$ and $\alpha(ur) = u\alpha(r)$ for all $r \in R$. If α is not inner of any periodicity, then it is said to be *outer*.

Remark 9.4. Notice that if α is an automorphism and u, v are as above, then v is the inverse of u.

In [4], the authors introduce a topology on $\mathcal{M}(R)$ in the following way. A net $(x_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathcal{M}(R)$ converges *strictly* to an element $x \in \mathcal{M}(R)$ if there for every $r \in R$ exists $\lambda_0 \in \Lambda$ such that $(x_\lambda - x)r = r(x_\lambda - x) = 0$ for $\lambda \geqslant \lambda_0$. Since R has local units, a net in $\mathcal{M}(R)$ can at most converge strictly to one element. Such an element will, if it exists, be called the *strict limit* of the net. A net $(x_\lambda)_{\lambda\in\Lambda}$ is *Cauchy* if there for every $r \in R$ exists $\lambda_0 \in \Lambda$ such that $r(x_\lambda - x_\mu) = (x_\lambda - x_\mu)r = 0$ for $\lambda, \mu \geqslant \lambda_0$. It is shown in [4, Proposition 1.6] that if R has local units, then every Cauchy net in $\mathcal{M}(R)$ converges strictly, and that every element of $\mathcal{M}(R)$ is the strict limit of a net of elements of R.

A net $(r_\lambda)_{\lambda\in\Lambda}$ of elements of R that converges to the unit of $\mathcal{M}(R)$ is called an *approximate unit* for R. Notice that in case R has local units we can construct an approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ consisting of idempotents simple by letting Λ be the directed set of finite subsets of R ordered by inclusion, and then for every $\lambda \in \Lambda$ choosing an idempotent e_λ such that $e_\lambda r = re_\lambda = r$ for every $r \in \lambda$.

Definition 9.5. Let R be a ring with local units. A ring homomorphism $\alpha : R \to R$ is said to be *strict* if there exists an approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ for R consisting of idempotents such that $(\alpha(e_\lambda))_{\lambda\in\Lambda}$ converges strictly.

Remark 9.6. Notice that if α is an automorphism, then it is strict (since $(\alpha(e_\lambda))_{\lambda\in\Lambda}$ converges strictly to the unit in that case). Notice also that if R is unital, then every ring homomorphism $\alpha : R \to R$ is automatically strict (because the net consisting of just 1 is an approximate unit in that case).

Proposition 9.7. *Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$, and let (P, Q, ψ) be the R-system defined above. Consider the following three conditions*:

(1) *There exists an $n \in \mathbb{N}$ such that the homomorphism α is inner with periodicity n.*
(2) *The ring R is a ψ-invariant cycle.*
(3) *The ring R does not satisfy condition* (L) *with respect to (P, Q, ψ).*

Then (1) *implies* (2)*, and* (2) *implies* (3)*. If in addition R is a super maximal ψ-compatible ideal, and α^n is strict for every $n \in \mathbb{N}$, then* (3) *implies* (1) *and the three conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): Let u and v be elements in $\mathcal{M}(R)$ such that $vu = 1$, and $urv = \alpha^n(r)$ and $\alpha(ux) = u\alpha(x)$ for all $r \in R$. Define $\eta : R \to R$ by $\eta(r) = ur$. Let $r \in R$. Choose $e \in R$ such that $er = r$. Then we have that $\eta(r) = ur = uer = uevur = \alpha^n(e)ur$. This shows that $\eta(R) \subseteq Q^{\otimes n}$. It is clear that η is additive and injective. Let $r_1, r_2, \in R$. Then $\eta(r_1r_2) = ur_1r_2 = \eta(r_1)r_2$ and $\eta(r_1r_2) = ur_1r_2 = \alpha^n(r_1)ur_2 = \alpha^n(r_1)\eta(r_2)$, which shows that η is an R-bimodule homomorphism from R to $Q^{\otimes n}$. Let $p \in P$, $r \in R$ and $q \in Q$. Then we have that

$$\begin{aligned}\eta\big(\psi(p \cdot r \otimes q)\big) &= \eta\big(p\alpha(r)q\big) = up\alpha(r)q = \alpha^n(p)u\alpha(r)q\\ &= \alpha^n(p)\alpha(ur)q = \alpha^n(p)\alpha\big(\eta(r)\big)q = S_pT_{\eta(r)}(q).\end{aligned}$$

Thus R is a ψ-invariant cycle.

(2) $\Rightarrow$ (3): It is easy to see that $\psi^{-1}(R) = R$ from which it follows that $R^{[\infty]} = R$. Thus, if R is a ψ-invariant cycle, then R does not satisfy condition (L) with respect to (P, Q, ψ).

(3) $\Rightarrow$ (1): Assume that R does not satisfy condition (L) with respect to (P, Q, ψ). It then follows from Proposition 3.8 that there is a non-zero graded ideal $\bigoplus_{k\in\mathbb{Z}} H^{(k)}$ in $\mathcal{O}_{(P,Q,\psi)}(R)$, an $n \in \mathbb{N}$ and a family $(\phi_k)_{k\in\mathbb{Z}}$ of injective $\mathcal{O}_{(P,Q,\psi)}(R)^{(0)}$-bimodule homomorphisms $\phi_k : H^{(k)} \to \mathcal{O}_{(P,Q,\psi)}(R)^{(k+n)}$ such that $x\phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ for $k, j \in \mathbb{Z}$, $x \in \mathcal{O}_{(P,Q,\psi)}(R)^{(j)}$ and $y \in H^{(k)}$. Notice that also $\bigoplus_{k\in\mathbb{Z}} \phi_{k-n}(H^{(k-n)})$ is a non-zero graded ideal in $\mathcal{O}_{(P,Q,\psi)}(R)$. If R is a super maximal ψ-compatible ideal, then it follows from Corollary 9.2 that

$$\bigoplus_{k\in\mathbb{Z}} H^{(k)} = \bigoplus_{k\in\mathbb{Z}} \phi_{k-n}\big(H^{(k-n)}\big) = \mathcal{O}_{(P,Q,\psi)}(R)$$

from which it follows that $H^{(0)} = \phi_{-n}(H^{(-n)}) = \mathcal{O}_{(P,Q,\psi)}(R)^{(0)} = \sigma(R)$, $\phi_0(H^{(0)}) = \mathcal{O}_{(P,Q,\psi)}(R)^{(n)} = T^n(Q^{\otimes n})$ and $H^{(-n)} = \mathcal{O}_{(P,Q,\psi)}(R)^{(-n)} = S^n(P^{\otimes n})$. Suppose in addition that α^n is strict, and let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for R consisting of idempotents such that $(\alpha(e_\lambda))_{\lambda\in\Lambda}$ converges strictly. Since T^n and ϕ_{-n} are injective, and $Q^{\otimes n}$ and $P^{\otimes n}$ are subsets of R, there exists for each $\lambda \in \Lambda$

a unique $u_\lambda \in R$ such that $T^n(u_\lambda) = \phi_0(\sigma(e_\lambda))$ and a unique $v_\lambda \in R$ such that $\phi_{-n}(S^n(v_\lambda)) = \sigma(e_\lambda)$. Notice that

$$T^n(u_\lambda) = \phi_0\big(\sigma(e_\lambda)\big) = \phi_0\big(\sigma(e_\lambda e_\lambda)\big) = \sigma(e_\lambda)\phi_0\big(\sigma(e_\lambda)\big) = \sigma(e_\lambda)T^n(u_\lambda) = T^n\big(\alpha^n(e_\lambda)u_\lambda\big).$$

It follows that $\alpha^n(e_\lambda)u_\lambda = u_\lambda$. If $\lambda, \lambda_1 \in \Lambda$ and $e_{\lambda_1}e_\lambda = e_{\lambda_1}$, then

$$\begin{aligned} T^n\big(\alpha^n(e_{\lambda_1})u_\lambda\big) &= \sigma(e_{\lambda_1})T^n(u_\lambda) = \sigma(e_{\lambda_1})\phi_0\big(\sigma(e_\lambda)\big) \\ &= \phi_0\big(\sigma(e_{\lambda_1}e_\lambda)\big) = \phi_0\big(\sigma(e_{\lambda_1})\big) = T^n(u_{\lambda_1}), \end{aligned}$$

from which it follows that $\alpha^n(e_{\lambda_1})u_\lambda = u_{\lambda_1}$. Let $r \in R$. Choose $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ such that $r\alpha^n(e_\lambda) = r\alpha^n(e_{\lambda_1})$ for $\lambda \geqslant \lambda_1$, $e_{\lambda_1}e_\lambda = e_{\lambda_1}$ for $\lambda \geqslant \lambda_2$, and $e_\lambda r = r$ for $\lambda \geqslant \lambda_3$. If $\lambda \geqslant \lambda_1, \lambda_2, \lambda_3$, then

$$ru_\lambda = r\alpha^n(e_\lambda)u_\lambda = r\alpha^n(e_{\lambda_1})u_\lambda = ru_{\lambda_1},$$

and

$$T^n(u_\lambda r) = T^n(u_\lambda)\sigma(r) = \phi_0\big(\sigma(e_\lambda)\big)\sigma(r) = \phi_0\big(\sigma(e_\lambda r)\big) = \phi_0\big(\sigma(r)\big).$$

This shows that $(u_\lambda)_{\lambda\in\Lambda}$ is Cauchy and hence converges strictly to an element $u \in \mathcal{M}(R)$. One can by a similar method show that $(v_\lambda)_{\lambda\in\Lambda}$ converges strictly to an element $v \in \mathcal{M}(R)$.

Let $\lambda \in \Lambda$. Then

$$\begin{aligned} \sigma(v_\lambda u_\lambda) &= S^n(v_\lambda)T^n(u_\lambda) = S^n(v_\lambda)\phi_0\big(\sigma(e_\lambda)\big) = \phi_{-n}\big(S^n(v_\lambda)\sigma(e_\lambda)\big) \\ &= \phi_{-n}\big(S^n(v_\lambda)\big)\sigma(e_\lambda) = \sigma(e_\lambda)\sigma(e_\lambda) = \sigma(e_\lambda), \end{aligned}$$

from which it follows that $v_\lambda u_\lambda = e_\lambda$. Thus $vu = 1$.

Let $r \in R$. Choose $\lambda_0 \in \Lambda$ such that $re_\lambda = e_\lambda r = r$ for $\lambda \geqslant \lambda_0$. If $\lambda \geqslant \lambda_0$, then

$$T^n\big(\alpha^n(r)u_\lambda\big) = \sigma(r)\phi_0\big(\sigma(e_\lambda)\big) = \phi_0\big(\sigma(re_\lambda)\big) = \phi_0\big(\sigma(e_\lambda r)\big) = \phi_0\big(\sigma(e_\lambda)\big)\sigma(r) = T^n(u_\lambda r).$$

It follows that $\alpha^n(r)u = ur$ and thus that $urv = \alpha^n(r)$.

Let $r \in R$. Choose $\lambda_0 \in \Lambda$ such that $e_\lambda r = r$ and $e_\lambda \alpha(r) = \alpha(r)$ for $\lambda \geqslant \lambda_0$. If $\lambda \geqslant \lambda_0$ then

$$\begin{aligned} T^n\big(\alpha(u_\lambda r)\big) &= T^n\big(\alpha^{n+1}(e_\lambda)\alpha(u_\lambda)\alpha(r)\big) = S\big(\alpha(e_\lambda)\big)T^n(u_\lambda)T\big(\alpha(r)\big) \\ &= S\big(\alpha(e_\lambda)\big)\phi_0\big(\sigma(e_\lambda)\big)T\big(\alpha(r)\big) = \phi_0\big(S\big(\alpha(e_\lambda)\big)\big)\sigma(e_\lambda)T\big(\alpha(r)\big) \\ &= \phi_0\big(\sigma\big(\alpha(e_\lambda e_\lambda r)\big)\big) = \phi_0\big(\sigma\big(\alpha(r)\big)\big) = \phi_0\big(\sigma\big(e_\lambda\alpha(r)\big)\big) = \phi_0\big(\sigma(e_\lambda)\big)\sigma\big(\alpha(r)\big) \\ &= T^n(u_\lambda)\sigma\big(\alpha(r)\big) = T^n\big(u_\lambda\alpha(r)\big), \end{aligned}$$

from which it follows that $\alpha(u_\lambda r) = u_\lambda\alpha(r)$. Thus $\alpha(ur) = u\alpha(r)$.

Hence α is inner with periodicity n in this case. □

By combining Theorem 7.3 and Corollary 9.2 with Remark 9.6, Proposition 9.7, and the fact that $\mathcal{O}_{(P,Q,\psi)}(R)$ is isomorphic to the crossed product $R \times_\alpha \mathbb{Z}$ of R by α when α is an automorphism, and to the fractional skew monoid ring $R[t_+, t_-; \alpha]$ when R is unital and α is an injective homomorphism such that $\alpha(R) = eRe$ for some idempotent $e \in R$, we get the following two corollaries.

Corollary 9.8. *Let R be a unital ring and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R) = eRe$ for some idempotent $e \in R$. Then the following two statements are equivalent:*

(1) *The fractional skew monoid ring $R[t_+, t_-; \alpha]$ is simple.*
(2) *The homomorphism α is outer and the only strongly α-invariant ideals in R are $\{0\}$ and R.*

Corollary 9.9. *Let R be a ring with local units and let $\alpha : R \to R$ be a ring automorphism. Then the following two statements are equivalent:*

(1) *The crossed product $R \times_\alpha \mathbb{Z}$ is simple.*
(2) *The automorphism α is outer and the only strongly α-invariant ideals in R are $\{0\}$ and R.*

We end with two simple examples which illustrate how Corollaries 9.8 and 9.9 can be used.

Example 9.10. Let F be a field, let R be the ring $\bigoplus_{n\in\mathbb{Z}} F$, and let $\alpha : R \to R$ be the ring automorphism given by $(\alpha(x))_n = x_{n+1}$ for $x = (x_k)_{k\in\mathbb{Z}} \in R$ and $n \in \mathbb{N}$. Since every ideal in R has the form $\{(x_k)_{k\in\mathbb{Z}} \in R \mid x_k = 0 \text{ for all } k \in A\}$ for some $A \subseteq \mathbb{Z}$, it follows that the only strongly α-invariant ideals in R are $\{0\}$ and R.

Suppose that there are an $n \in \mathbb{N}$ and $u, v \in \mathcal{M}(R)$ such that $vu = 1$ and $\alpha^n(x) = uxv$ for all $x \in R$. Since R is commutative, it follows from [4, Lemma 1.3] and the fact that every element of $\mathcal{M}(R)$ is the strict limit of a net of elements of R, that $\mathcal{M}(R)$ is also commutative. Thus $\alpha^n(x) = x$ for every $x \in R$. This cannot be the case, so α is outer. It follows from Corollary 9.9 that the crossed product $R \times_\alpha \mathbb{Z}$ is simple.

Notice that the crossed product $R \times_\alpha \mathbb{Z}$ can be identified with the ring generated by elements $\{[y, m, n] \mid y \in F,\ m, n \in \mathbb{Z}\}$ satisfying

$$[y_1, m, n] + [y_1, m, n] = [y_1 + y_2, m, n]$$

for $y_1, y_2 \in F$ and $m, n \in \mathbb{Z}$,

$$[y_1, m_1, n_1][y_2, m_2, n_2] = \begin{cases} [y_1 y_2, m_1, n_1 + n_2] & \text{if } m_1 = m_2 - n_1, \\ 0 & \text{if } m_1 \neq m_2 - n_1, \end{cases}$$

for $y_1, y_2 \in F$ and $m_1, n_1, m_2, n_2 \in \mathbb{Z}$.

Example 9.11. Let F be a field, n an integer greater than 1 and let R be the inductive limit of $\bigotimes_{i=1}^k M_n(F)$ where $M_n(F)$ denotes the ring of $n \times n$-matrices over F and the transition map from $\bigotimes_{i=1}^k M_n(F)$ to $\bigotimes_{i=1}^{k+1} M_n(F)$ is given by $x_1 \otimes \cdots \otimes x_k \mapsto 1 \otimes x_1 \otimes \cdots \otimes x_k$ for $x_1, \ldots, x_k \in M_n(F)$. Then R is a unital ring.

Let $(e_{i,j})_{i,j=1}^n$ denote the standard matrix units in $M_n(F)$. For each $k \in \mathbb{N}$ let α_k be the ring homomorphism from $\bigotimes_{i=1}^k M_n(F)$ to $\bigotimes_{i=1}^{k+1} M_n(F)$ given by $\alpha(x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_k \otimes e_{1,1}$. Then $(\alpha_k)_{k\in\mathbb{N}}$ induces an injective ring homomorphism $\alpha : R \to R$ such that $\alpha(R) = eRe$ where e is the image of $e_{1,1}$ in R. Notice that e is an idempotent. It is shown in [3, Example 2.5] that the fractional skew monoid ring $R[t_+, t_-; \alpha]$ is isomorphic to Leavitt algebra $V_{1,n}(F)$ first studied in [10].

Suppose that there are an $m \in \mathbb{N}$ and $u, v \in R$ such that $vu = 1$ and $\alpha^m(x) = uxv$ for all $x \in R$. Choose $k \in \mathbb{N}$ such that u and v belong to the image of $\bigotimes_{i=1}^k M_n(F)$ in R. Then $\alpha^m(x) = uxv$ belongs to the image of $\bigotimes_{i=1}^k M_n(F)$ in R whenever x does. That cannot be the case, so α is outer. Since R is simple, the only strongly α-invariant ideals in R are $\{0\}$ and R. It therefore follows from Corollary 9.8 that the fractional skew monoid ring $R[t_+, t_-; \alpha]$ is simple.

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